PROJECTILE MOTION

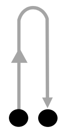
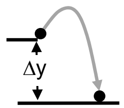
VERTICAL PROJECTILE MOTION (1 DIMENSION)

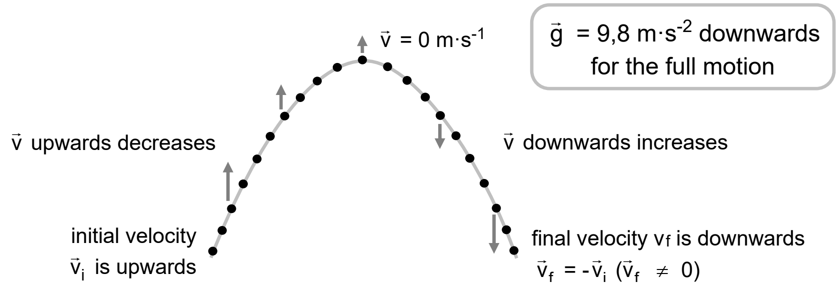
Definitions

The following terms are important in vertical projectile motion:

- **Gravitational field:** This is the space around an object of mass (e.g. the earth) in which another mass experiences a gravitational force.
- **Acceleration due to gravity (g):** The acceleration of a falling object as a result of the attractive gravitational force of the earth, in the absence of other forces such as air resistance.
- **Free fall:** The unhindered movement of an object in the gravitational field of the earth, when only the gravitational force is acting on it.
- **Projectile:** An object that is moving through the air, experiencing free fall, after being dropped, thrown or shot. (In the air, air resistance is small. If air resistance is ignored, gravitational force is the only force acting on the object.)

Representation of three different types of projectile motion

❶ Vertical downward projection from a starting point above the ground. The object can: › be dropped ($v_i = 0$) **OR** › be thrown downwards at a velocity $v_i > 0$. 	❷ Vertical upward projection from a starting point; object turns around and returns to starting point.  $v_i \neq 0$	❸ Vertical upward projection from a starting point above the ground; object turns, moves downwards, past the starting point, to the ground. 



- **Time symmetry:** The time interval from a certain height to the turning point = time interval from the turning point to the same height.
- At the **same time** interval Δt on either side of the turning point, the instantaneous velocity is the same, but in the opposite direction.
- The **displacement** ($\Delta\vec{y}$) is the linear distance from an initial position ($\vec{y}_i$) to a final position ($\vec{y}_f$) for the specific part of the motion that is being worked with.
- The displacement ($\Delta\vec{y}$) can be upwards or downwards with respect to the initial position. Use the correct sign for $\Delta\vec{y}$ according to how the direction was chosen, i.e. ↑+/↓- or vice versa. When returned to the starting point, $\Delta\vec{y} = y_f - y_i = 0$.

Description of projectile motion

Describe projectile motion in terms of **words**, **equations** and **graphs**:

- **In words:** Give a description in words of the object's position, initial and final velocities, acceleration, displacement and the duration of time of the motion.
- **Equations of motion:** A projectile motion upwards and downwards can be described by a single set of equations, namely one of the following equations of motion:

Equations of motion

Horizontal motion	Vertical motion
$v_f = v_i + a\Delta t$	$v_f = v_i + a\Delta t$
$\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a}\Delta x$	$\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a}\Delta y$
$\Delta\vec{x} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$	$\Delta\vec{y} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$
$\Delta\vec{x} = \left(\frac{\vec{v}_i + \vec{v}_f}{2}\right)\Delta t$	$\Delta\vec{y} = \left(\frac{\vec{v}_i + \vec{v}_f}{2}\right)\Delta t$

where:
$\vec{v}_i$ = initial velocity ($m \cdot s^{-1}$)
$\vec{v}_f$ = final velocity ($m \cdot s^{-1}$)
$\vec{a}$ = acceleration ($m \cdot s^{-2}$)
$\vec{g}$ = gravitational acceleration = $9{,}8\ m \cdot s^{-2}$
Δt = time (s)
$\Delta\vec{x}$; $\Delta\vec{y}$ = displacement (m) ($\Delta\vec{x}$ for horizontal, $\Delta\vec{y}$ for vertical)

Application: Problem-solving steps

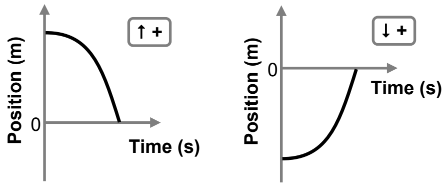
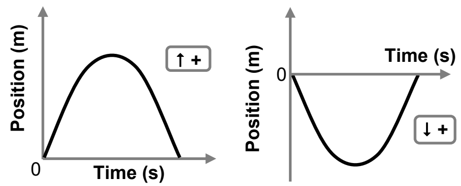
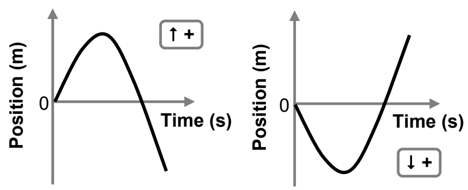
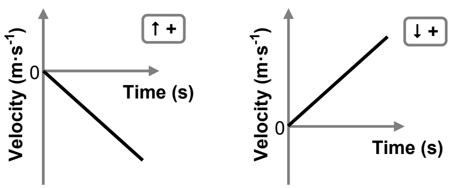
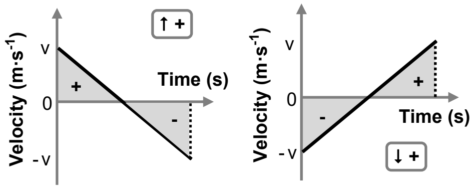
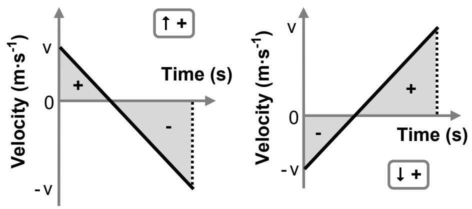
- Read the problem carefully and draw a sketch with all the necessary information.
- Determine with which part of the motion a specific question is concerned, e.g. with the full motion or only the upward motion up to the turning point, or with a certain time interval, etc.
- Choose the direction of motion if working with both upward and downward motion in one calculation, e.g. if upwards is chosen as **+** , downwards is taken as **-**.
  - Every equation of motion contains four of the five possible variables, i.e. $\vec{v}_i$, $\vec{v}_f$, $\vec{g}$, Δt, $\Delta\vec{y}$.
  - Make a list of all the variables and fill in the values provided.

Remember:
The value of a variable can differ for different parts of the motion, e.g. velocity at turning point is the v_f for ↑ and v_i for ↓ motion.

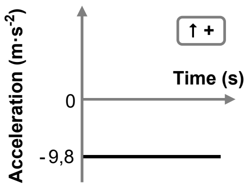
  - $\vec{g} = 9{,}8\ m \cdot s^{-2}$ is known already; therefore we only need the values of any two other variables in order to calculate the missing value in an equation.
  - Choose the correct equation.
  - Remember the signs of $\vec{v}_i$, $\vec{v}_f$ and $\Delta\vec{y}$.
  - The sign for $\vec{g}$ will be the same as that of the downward motion.
- Do substitution in the applicable equation.
- Do the calculation and give the answer with the correct unit and direction.

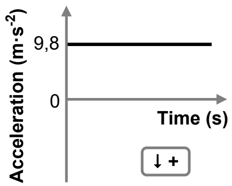
Remember:
- *$\vec{v}_i$ = $0\ m \cdot s^{-1}$ if the object falls from rest* ***OR***
- *$\vec{v}_i$ = the velocity of another moving object off which the object falls*
- *the velocity is $0\ m \cdot s^{-1}$ at the turning point*
- *the velocity at which the object falls increases during downward free fall. The velocity at which it hits the ground ($\vec{v}_f$) $\neq 0$, but it has a maximum value.*

Graphs of motion for vertical motion

Vertical downward projectile motion	Projectile motion up- and downward, BACK TO STARTING POSITION	Projectile motion up- and downward, PAST STARTING POSITION
Position-time  The ground is taken as reference point (zero)	**Position-time**  The starting position was taken as reference point (zero).	**Position-time**  The starting position was taken as reference point (zero).
	 ***NB:*** *If the turning point or end position is taken as the reference point, the entire graph moves up or down to that position.*	
Velocity-time  Gradient $= \frac{\Delta \vec{v}}{\Delta t}$ $= \vec{a}$ $= -9{,}8\ m \cdot s^{-2}$ / $+9{,}8\ m \cdot s^{-2}$ $\Delta \vec{y}$ = area under graph = area of Δ $= \frac{1}{2}bh$	**Velocity-time**  › Gradient $= \frac{\Delta \vec{v}}{\Delta t}$ $= \vec{a} = -9{,}8\ m \cdot s^{-2}$ / $+9{,}8\ m \cdot s^{-2}$ › $\Delta \vec{y}$ = area under graph = sum of the triangles between graph and horizontal axis = 0 (the triangles are the same size, but in opposite directions; therefore they cancel one another out)	**Velocity-time**  › Gradient $= \frac{\Delta \vec{v}}{\Delta t}$ $= \vec{a} = -9{,}8\ m \cdot s^{-2}$ / $+9{,}8\ m \cdot s^{-2}$ › $\Delta \vec{y}$ = area under graph = sum of the two triangles between the graph and horizontal axis (the total displacement has the same sign as the larger triangle)

Acceleration-time
(for all three cases)

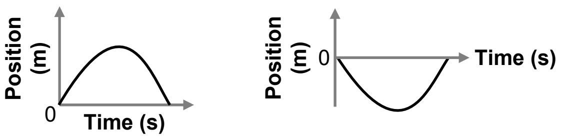


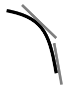


SINGLE PROJECTILE MOTION

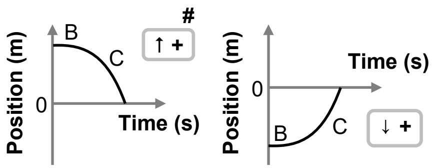
Position-time graph (for uniform acceleration)

The position-time graph for an upward and downward projectile motion has a parabolic shape.

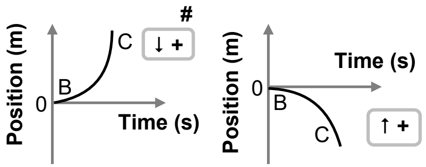


- The zero position is the reference point relative to which the object moves, e.g. the ground, the top of a building, etc.
- If the graph moves further away from the reference point (zero position), the displacement increases, and if it moves closer to the reference point, the displacement decreases.
- The gradient of the tangent to the graph at any point gives the instantaneous velocity of the object at that moment:
  -  A decreasing gradient shows that the velocity is busy decreasing, e.g. during upward motion.
  -  An increasing gradient shows that the velocity is busy increasing, e.g. during downward motion.

❶ Object falls downwards to/from the reference point

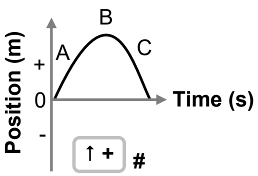


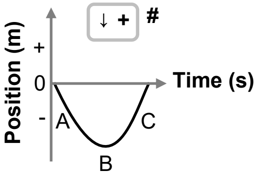
Ground as reference point



Starting position as reference point

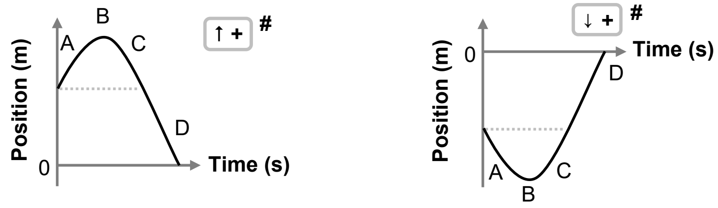
❷ Object is projected upwards and returns to the initial position





Starting position (ground) as reference point

❸ Object is projected upwards and falls past the initial position



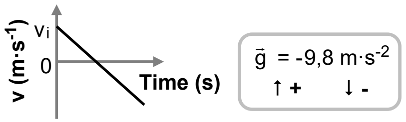
(ground as reference point)

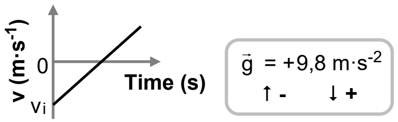
(a position above the ground is the reference point)

 # Note: *For position-time graphs:* ↑ = *position* ***above*** *reference point*
(*indication of direction*) ↓ = *position* ***below*** *reference point*

Velocity-time graph (for uniform acceleration)

- The velocity-time graph for an object accelerating uniformly is a straight line.
- The gradient of the graph gives the acceleration, i.e. 9,8 m·s^{-2} downwards. The sign of the gradient therefore indicates the sign of the downward motion.

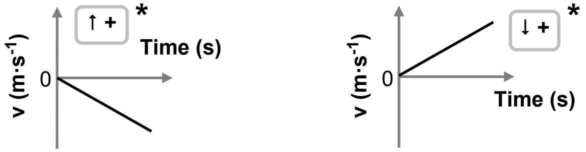




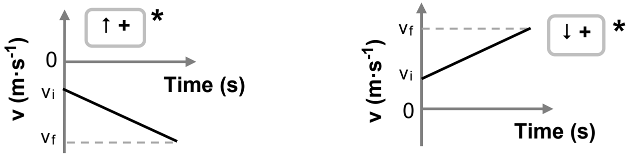
- The initial velocity is read on the vertical axis opposite t = 0 (y-intercept)
- The point of intersection with the time axis is the turning point in the air where the velocity = 0 m·s^{-1}.

- If the graph starts on the time axis, e.g. $\vec{v} = 0$, it falls downwards from rest. The motion is only in one direction if the object falls or is thrown downwards, e.g.:

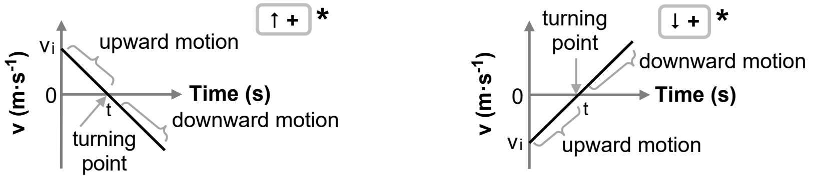
Object falls from rest ($v_i = 0\ m{\cdot}s^{-1}$)



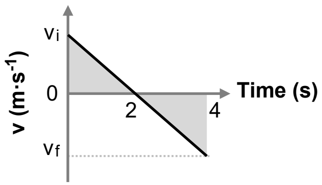
Object thrown downwards



- If the graph intersects the time axis, the object changes direction at time t. The object first moves upwards (velocity gradually decreases to zero at the turning point) and afterwards moves downwards with increasing velocity.



- The area between the graph and the horizontal axis indicates the displacement of the object at any given moment.



after 2 s:

$\Delta y = \frac{1}{2}bh$

$= \frac{1}{2}(2)v_i$

after 4 s:

$\Delta y = \frac{1}{2}b_1h_1 + \frac{1}{2}b_2h_2$

$= \frac{1}{2}(2)v_i + \frac{1}{2}(2)(-v_i)$ ($v_f = -v_i$)

$= 0$ m

after 4 s: time symmetry ∴ $\vec{v}_f = -\vec{v}_i$ and $\Delta\vec{y} = 0$ m

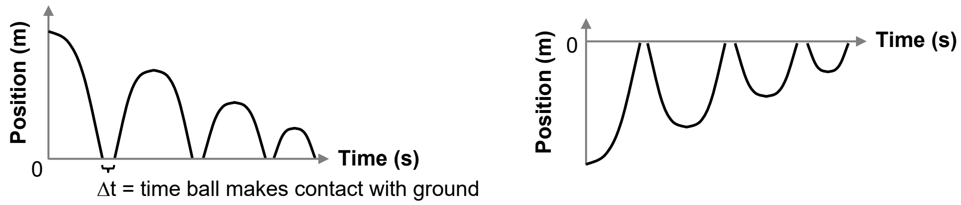
*** *Note:*** *For velocity-time graphs:* ↑ = ***upward*** *motion*
(*indication of direction*) ↓ = ***downward*** *motion*

THE BOUNCING BALL

Ball is dropped from rest at a certain height and bounces on the ground a few times.

❶ Position-time graph:

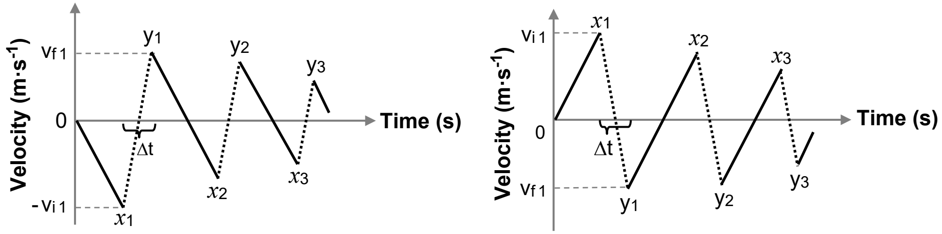
i) ↑ + ii) ↓ +



Each bounce is a separate projectile motion of an object, which is projected upwards and returns to the starting position.

❷ Velocity-time graph:

i) ↑ + ii) ↓ +

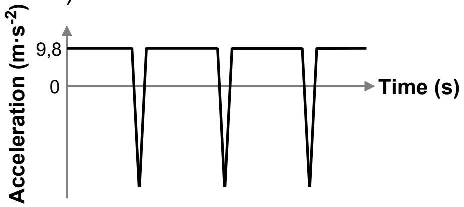


- The ball hits the ground at x_1, x_2, x_3 (bounces three times).
- It is in contact with the ground for a very short time interval Δt.
- It bounces back and leaves the ground in the opposite direction at points y_1, y_2, and y_3.
- The turning point in the air for each bounce is where the graph intercepts the time axis (v = 0).
- The lines are parallel. The gradient of the lines indicates the acceleration of the ball, i.e. gravitational acceleration (gradient = $-9{,}8\ m{\cdot}s^{-2}$ for ↑**+** and $+9{,}8\ m{\cdot}s^{-2}$ for ↓**+**).

❸ Acceleration-time graph:

i) ↑ +

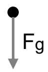
ii) ↓ +



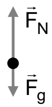
The ball experiences a constant gravitational acceleration of 9,8 m·s^{-2} ↓. The sharp points on the graph are the large upward accelerations (due to net force ↑), when the ball is in touch with the ground for a short time interval.

Please note

- When a ball moves through the air (free-fall), the only force acting on it is gravitational force ($\vec{F}_g$).



- The ball is only a projectile as it moves through the air, not when it touches the ground.
- When the ball hits the ground, the ground exerts an upward force ($\vec{F}_N$) on the ball for a short time interval (Δt), there is a resultant/net force, $\vec{F}_{net}$ > 0, on the ball.



$\vec{F}_{net} = \vec{F}_N + \vec{F}_g$ (vector addition)
upwards

- The ball has a maximum velocity and kinetic energy E_k just before it hits the ground.
- During each bounce, some of the E_k is converted into other forms of energy.
- The ball will therefore bounce back with a smaller magnitude of velocity than that with which it hit the ground. It will therefore bounce back to a lower height than with the previous bounce.
- Each bounce is a separate projectile motion. Work with each bounce/projectile motion separately to solve problems.

Remember:

A ball experiences an impulse and undergoes a change in momentum (Δp) when it hits the ground.

- *Impulse* $= \vec{F}_{net}\Delta t = \Delta\vec{p}$ ➡ $\therefore \vec{F}_{net} = \frac{\Delta\vec{p}}{\Delta t}$

  The smaller Δt, the larger $\vec{F}_{net}$ for the same $\Delta\vec{p}$.
- $\Delta\vec{p} = m(\vec{v}_f - \vec{v}_i)$ *(where $\vec{v}_f$ = velocity with which the ball leaves the ground /bounces back and $\vec{v}_i$ = velocity with which the ball hits the ground)*

PROJECTILE MOTION

❶ Definitions

- **Free fall**
- **Gravitational acceleration** ($\vec{g}$ = 9,8 m·s^{-2} downward)
- **Projectile**

❷ Downward ↓

❸ Up- and downward; back to starting position

❹ Up- and downward; past starting position

❺ Bouncing ball

- dropped/thrown vertically downward **or** thrown upward; bounces on the ground several times
- each bounce is:
  - an inelastic collision
  - a separate projectile motion

Describe (in words)

- constant acceleration i.e. $\vec{g}$ = 9,8 m·s^{-2} downward
- **for ↑ motion:** velocity decreases to 0 m·s^{-1} at turning point
- **for ↓ motion:** velocity increases up to a maximum just before it hits the ground

Calculate

Choose direction, e.g. ↑+ and ↓-

- Use **equations of motion** and calculate, for example:
  - maximum **height** reached
  - height above ground from which object is projected
  - **time taken** to maximum height **or** until ground is reached
  - **velocity** at a certain time, etc.

Graphs

- position/displacement-time (Δy-t)
- velocity-time (v-t)
- acceleration-time (a-t)

Calculations: (v-t graph)

- gradient (m) = $\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$
- area under graph = displacement

EXAMPLES

Question 1

A 20 kg concrete block falls off a scaffolding on a building site, hits a pile of sand and comes to a standstill within 10 ms. The average net force of the sand and other forces on the concrete block is 10 500 N upwards. Take upwards **+**.

1.1 Calculate the impulse of the net force on the concrete block.

1.2 Calculate the velocity at which the block hits the sand before it comes to a standstill.

1.3 Calculate the height of the scaffolding.

1.4 Calculate the total time from when the block starts to fall until it comes to a standstill.

ANSWER

Take upwards as **+**

$m = 20$ kg
$\Delta t = 10$ ms $= 0{,}010$ s
$F_{net} = 10\ 500$ N upwards

1.1 impulse $= F_{net}\Delta t$
$= (10\ 500)(0{,}01)$
$= 105$ N·s upwards

During contact with ground:
$\Delta p = 105$ kg·m·s^{-1} upwards
$v_f = 0$ m·s^{-1}
$v_i = ?$
$\Delta t = 0{,}01$ s

1.2 impulse $= \Delta p = m(v_f - v_i)$
$\therefore 105 = (20)(0 - v_i)$
$= -20v_i$
$\therefore v_i = \frac{105}{-20}$
$= -5{,}25$ m·s^{-1}
$= 5{,}25$ m·s^{-1} downwards

For motion through the air:
$v_i = 0$ m·s^{-1}
$v_f = 5{,}25$ m·s^{-1} downwards
$= -5{,}25$ m·s^{-1}
$g = -9{,}8$ m·s^{-2}
$\Delta y = ?$
$\Delta t = ?$

1.3 $v_f^2 = v_i^2 + 2a\Delta y$
$(-5{,}25)^2 = 0 + 2(-9{,}8)(\Delta y)$
$19{,}6\Delta y = -27{,}5625$
$\Delta y = -1{,}41$ m
$= 1{,}41$ m downwards

1.4 $v_f = v_i + g\Delta t$
$\therefore -5{,}25 = 0 + (-9{,}8)\Delta t$
$\therefore \Delta t = 0{,}54$ s
$\therefore$ total time until it comes to a standstill $= 0{,}54 + 0{,}01$
$= 0{,}55$ s

Question 2

A cannonball is fired at a velocity of 25 m·s^{-1}. Calculate:

2.1 the maximum height that it reaches.

2.2 the time elapsed until it returns to the same horizontal level as the top end of the cannon barrel, from where it was fired.

2.3 the height above the ground at which the cannonball has an upwards velocity of 10 m·s^{-1} (assume the top of the barrel is 1 m above the ground).

ANSWER

Take upwards **+**
$v_i = 25$ m·s^{-1}
$v_f = 0$ m·s^{-1}
$g = -9{,}8$ m·s^{-2}
$\Delta y = ?$

2.1 $v_f^2 = v_i^2 + 2g\Delta y$
$\therefore 0 = 25^2 + 2(-9{,}8)\Delta y$
$\therefore 19{,}6\Delta y = 625$
$\therefore \Delta y = 31{,}89$ m

2.2 $v_f = v_i + g\Delta t$
$\therefore 0 = 25 + (-9{,}8)\Delta t$
$\therefore \Delta t = \frac{25}{9{,}8}$
$= 2{,}55$ s
$\therefore$ total time $= (2{,}55)(2) = 5{,}10$ s

$v_i = 25$ m·s^{-1} upwards
$v_f = 10$ m·s^{-1} upwards
$\Delta y = ?$

2.3 $v_f^2 = v_i^2 + 2g\Delta y$
$\therefore 10^2 = 25^2 + 2(-9{,}8)\Delta y$
$\therefore 19{,}6\Delta y = 625 - 100$
$= 525$
$\therefore \Delta y = 26{,}79$ m
$= 26{,}79 + 1 = 27{,}79$ m above the ground

Question 3

A hot-air balloon travels upwards at a constant velocity. At a height of 80 m above the ground, someone drops his cell phone from the balloon. It hits the ground after 5 seconds. Ignore the effect of air resistance.

3.1 Calculate the velocity at which the balloon travels upwards.

3.2 Calculate the maximum height above the ground reached by the cell phone.

3.3 How long after the cell phone has been dropped, does it pass the point from where it started to fall?

3.4 How long after the cell phone has been dropped, does the distance between the balloon and the cell phone equal 20 m?

ANSWER

3.1 $\Delta y = v_i \Delta t + \frac{1}{2} g \Delta t^2$

$\therefore -80 = v_i(5) + \frac{1}{2}(-9,8)(5)^2$

$= 5v_i - 122,5$

$\therefore 5v_i = 122,5 - 80$

$= 42,5$

$\therefore v_i = 8,5 \text{ m·s}^{-1}$ upwards

> Take upwards as **+**
> v_i (cellphone) = velocity of balloon
> $\Delta y = -80$ m
> $\Delta t = 5$ s
> $g = -9,8 \text{ m·s}^{-2}$

3.2 $v_f^2 = v_i^2 + 2g\Delta y$

$\therefore 0 = 8,5^2 + 2(-9,8)\Delta y$

$\therefore 19,6\Delta y = 72,25$

$\therefore \Delta y = 3,69$ m

maximum height above the ground = 80 + 3,69 = 83,69 m

> Take upwards as **+**
> For ↑ motion to turning point:
> $v_f = 0 \text{ m·s}^{-1}$
> $v_i = 8,5 \text{ m·s}^{-1}$ upwards
> $g = -9,8 \text{ m·s}^{-2}$
> $\Delta y = ?$

3.3 $v_f = v_i + g\Delta t$

$\therefore -8,5 = 8,5 + (-9,8)\Delta t$

$\therefore 9,8\Delta t = 17$

$\therefore \Delta t = 1,75$ s

> Velocity back at starting point $= -8,5 \text{ m·s}^{-1}$ (time symmetry)

3.4 Balloon moves upwards at a constant velocity:

$\therefore$ its displacement after t seconds: $\Delta y_1 = v\Delta t = 8,5\Delta t$

Cell phone falls at gravitational acceleration:

$\therefore$ its displacement after t seconds: $\Delta y_2 = v_i \Delta t + \frac{1}{2} g\Delta t^2$

$= 8,5\Delta t + \frac{1}{2}(-9,8)\Delta t^2$

$= 8,5\Delta t - 4,9\Delta t^2$

$\therefore$ Difference in distance $= \Delta y_1 - \Delta y_2$

$\therefore 20 = 8,5\Delta t - (8,5\Delta t - 4,9\Delta t^2)$

$= 4,9\Delta t^2$

$\therefore \Delta t^2 = 4,08$

$\therefore \Delta t = 2,02$ s

Question 4

Draw both a position-time and velocity-time graph for each of the following situations:

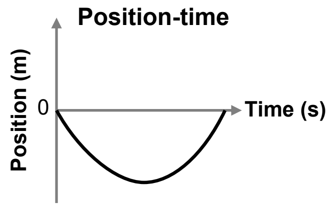
4.1 A ball is projected vertically upwards and returns to the hand of the thrower. Take downwards as positive.

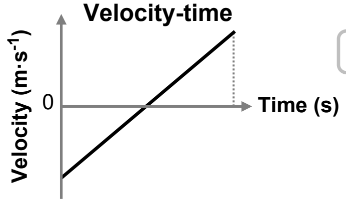
4.2 A stone is dropped from the top of a building and hits the ground after 2 s. Take downwards as positive.

4.3 A small rubber ball is dropped from a certain height, bounces twice on the ground and is caught again at its maximum height. Assume that its collision with the ground is inelastic. Take upwards as positive.

ANSWER

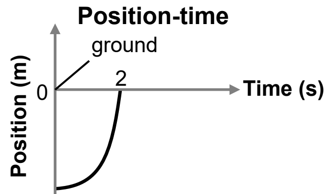
4.1

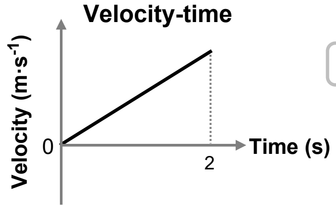




↓ +

4.2





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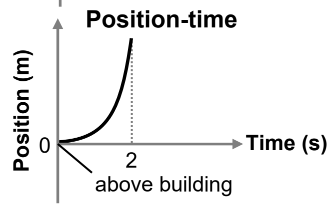
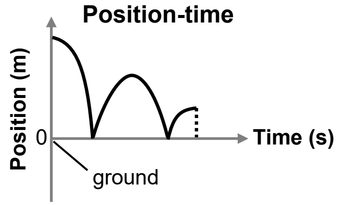
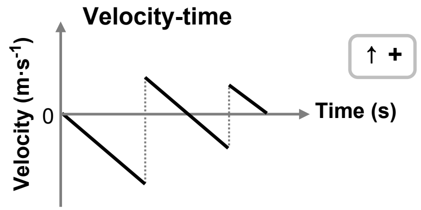


4.3

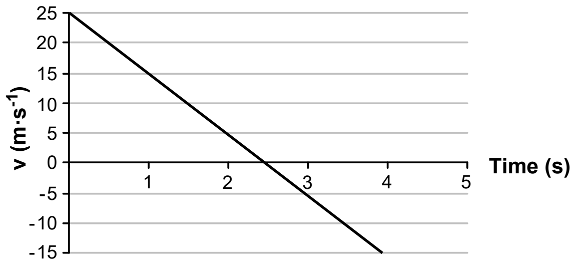




↑ +

Question 5

Consider the following velocity-time graph:



5.1 What is the initial velocity of the object? Assume upwards is positive.

5.2 How long does the object take to reach its maximum height?

5.3 At what time is the speed of the object 10 m·s^{-1}?

5.4 Determine the position of the object relative to the ground at t = 2 s and at t = 4 s.

5.5 Draw a position-time graph for the motion of the ball during the first 4 s.

ANSWER

5.1 $v_i = 25$ m·s^{-1} upwards

5.2 $\Delta t = 2{,}5$ s to turning point

5.3 at t = 1,5 s and again at t = 3,5 s

5.4 At t = 2 s, v = 5 m·s^{-1}:

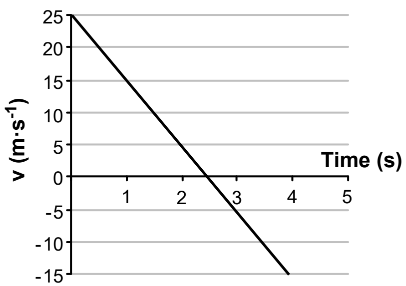
displacement = area under graph

$= \frac{1}{2}bh + (\ell b)$

$= \frac{1}{2}(2)(20) + (5)(2)$

$= 20 + 10$

= 30 m upwards



At t = 4 s, v = -15 m·s^{-1}:

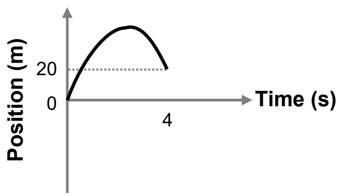
displacement = area under graph

$= \frac{1}{2}b_1h_1 + \frac{1}{2}b_2h_2$

$= \frac{1}{2}(2{,}5)(25) + \frac{1}{2}(1{,}5)(-15)$

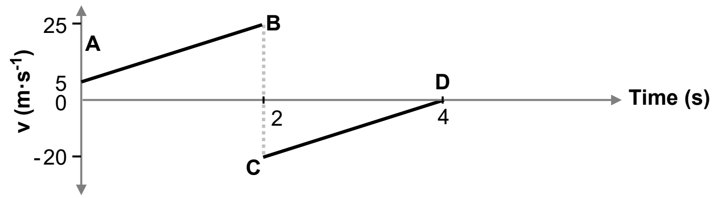
= 31,25 - 11,25 = 20 m upwards

5.5



Question 6

The following graph shows the velocity of a ball thrown vertically downwards from a height of x metres above the ground.



6.1 What is the initial velocity of the ball?

6.2 What is the acceleration of the ball as it falls to the ground?

6.3 Determine the value of x.

6.4 Describe the motion of the ball for sections AB and CD of the graph.

6.5 Draw the corresponding position-time graph of the motion of the ball.

ANSWER

6.1 5 m·s^{-1} downwards

6.2 acceleration = gradient

$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

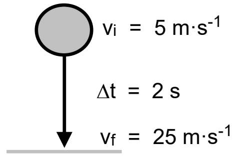
↓ +

$= \frac{25 - 5}{2 - 0}$

$= \frac{20}{2}$

= 10 m·s^{-2} downwards

6.3



area = rectangle + triangle

area $= (\ell b) + \left(\frac{1}{2}bh\right)$

$= (5)(2) + \frac{1}{2}(2)(25 - 5)$

$= 10 + 20 = 30$ m

OR

$v_f^2 = v_i^2 + 2a\Delta y$

$\therefore 25^2 = 5^2 + 2(10)\Delta y$

$\therefore 20\Delta y = 600$

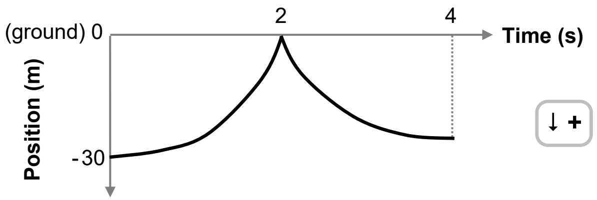
$\Delta y = 30$ m

ANSWER

6.4 AB: The ball is projected vertically downwards at a velocity of 5 m·s^{-1}. It accelerates downwards at 10 m·s^{-2} and hits the ground after 2 seconds with a velocity of 25 m·s^{-1}.

CD: The ball bounces back from the ground at a velocity of 20 m·s^{-1} and moves upwards at the same acceleration (10 m·s^{-2} downwards), to its turning point in the air. It reached a velocity of 0 m·s^{-1} at point D.

6.5



NB:

The ball initially moved downwards (started at a negative position above the ground and moved closer to the ground (zero position)). Thereafter the ball moved upwards (in the negative direction) to a lower height from which it was thrown.

NOTES