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DBE

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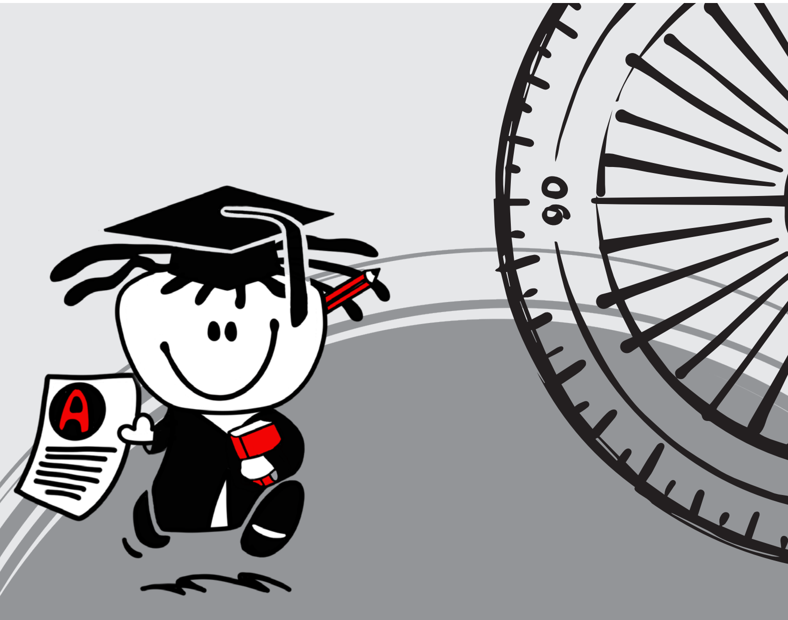
OFFICIAL EXAMS & MEMOS

Anne Eadie, Gretel Lampe, Jenny Campbell & Susan Carletti

GRADE

12

CAPS



Grade 12 Maths Toolkit | DBE Past Papers

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The Answer Series Grade 12 Maths Toolkit is a low-priced product, offering both theory and practice, and is perfect for exam preparation for matrics.

This **UP-TO-DATE** publication is indeed a TOOLKIT, containing:

- **DBE Nov Paper 1 & Paper 2 Exams** (2016 – 2023) with **comprehensive solutions** to all papers, including TOPIC GUIDES that make it possible to select questions on **separate topics**, as well as **challenging questions** from all these exams and totally aligned with DBE Diagnostic Reports since 2016.

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How learners can improve their exam techniques:

- write a few of the papers under exam conditions
- get comfortable with having to concentrate for the full 3 hour time period
- learn to work though the paper a few times, answering all the routine questions first, then coming back for more challenging questions that take more time, and
- finally, when all else is done, tackling the questions that need more time and attention

Good exam technique makes a huge difference to anyone's ability to produce top quality work under pressure and there is no doubt that The Answer Series Grade 12 Maths Toolkit levels the playing fields and ensures that everyone has equal access to success.

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NOTE

The questions marked with an asterisk (*) are identified as challenging . . . average performance < 40%.

VALUABLE DOCUMENTS

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Be sure to incorporate these Theory documents regularly as you revise.

We are grateful to the Department of Basic Education for granting their permission for the inclusion of these exam papers.

DBE P1: TOPIC GUIDE	2016	2017	2018	2019	2020	2021	2022	2023
➤ **Algebra: [25]**								
Quadratic equations & theory	1.1.1, 1.1.2, 1.2.1	1.1.1, 1.1.2	1.1.1, 1.1.2	1.1.1, 1.1.2	1.1.1, 1.1.2	1.1.1, 1.1.2, 1.3*	1.1.1, 1.1.2	1.1.1, 1.1.2
Quadratic inequalities	1.2.2	1.3.1	1.1.3	1.1.3	1.1.3	1.1.3	1.1.3	1.1.4
Simultaneous equations	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2
➤ **Exponents:**								
Expressions			1.3*				1.3	
Equations & Inequalities	1.1.4			1.3*	1.3*		1.4*	1.3*
➤ **Surds:**								
Expressions								
Equations	1.1.3	1.1.3	1.1.4	1.1.4	1.1.4	1.1.4	1.1.4	1.1.3
➤ **Logs (Application)**								
➤ **Patterns & Sequences: [25]**								
Quadratic	3.1*	2.1		2.1	2.2	3.1, 3.2, 3.3*, 3.4*	3	2.2
Arithmetic	2.1 – 2.3, 2.4*	2.2			2.1	4.1, 4.2		2.1
Geometric	3.2*		3.1, 3.2	2.2	11.3*	2	2.1	3.1.1
Σ		3*	3.3, 3.4*	3.1*	3.1, 3.2*	4.3*, 4.4	2.2	3.1.2
Mixed / General			2.1 – 2.3	3.2				3.2*
➤ **Finance, Growth & Decay: [15]**								
Simple & compound growth & decay		6.1		6.1	6.2	8.1, 8.2	6.1	6.1.1, 6.2.1
Nominal & Effective interest rates								6.1.2
Annuities	7.1 – 7.3, 7.4*	6.2*	7.2	6.2	6.1, 6.3*	8.3*	6.2, 6.3*	6.2.2, 6.3*
Time line			7.1*					
➤ **Functions & Graphs: [35]**								
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Exponent. & log function (incl. Inverses)	4.1 – 4.4, 4.5*					6.1, 6.2*, 6.3, 6.4*		
Inverse functions			4.1 – 4.3, 4.4*	5.1 – 5.3, 5.4*, 5.5*	5.1, 5.2, 5.3*, 5.4*, 5.5		5.1 – 5.4, 5.5*	
Mixed	5.1, 5.2*, 5.3, 5.4*, 5.5*, 6.1, 6.2, 6.3*, 6.4	5.1 – 5.5, 5.6*		4.1 – 4.6, 4.7*	4.2		4.2	4.1 – 4.7 (4.6*), 5*
➤ **Differential Calculus: [35]**								
Finding the derivative: 1st principles	8.1, 8.2*	7.1	8.1	7.1	7.1	9.1	7.1	7.1
Finding the derivative: using the rules	8.3	7.2	8.2	7.2, 7.3	7.2, 8.4	9.2	7.2	7.2
Finding the average gradient								
Tangent: the gradient & the equation	8.4*			7.4*				7.3*
Curve sketching & f'' & concavity	5.6*, 9.1, (9.2 – 9.4)*	8*	5.4*, 6.5*, 9.1*, 9.2	9.1, 9.2*, 9.3, 9.4*	8.1, 8.2, 8.3*	(10.1 – 10.4*)*	7.3*, 8.1, 8.2, 8.3*	8.1 – 8.5, (8.3)*
Practical application (incl. max/min)	1.2.3, 10.1, 10.2*, 10.3*	9*	10*	8.1, 8.2*, 8.3	8.5*, 9.1*, 9.2*	11*	9*	9*
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2-way Contingency tables	11.1, 11.2*, 11.3							
Fundamental Counting Principle	12*	11*	11*	10*, 11.2	10*	12.2*	10.2*	10.3*

Questions marked with an asterisk (*) were identified as challenging . . . ave performance < 40%

DBE NOV 2016 PAPER 1

Clearly show ALL calculations, diagrams, graphs, etc. that you have used in determining your answers.

Answers only will NOT necessarily be awarded full marks.

You may use an approved scientific calculator (non-programmable and non-graphical), unless stated otherwise.

*If necessary, round off answers to **TWO** decimal places, unless stated otherwise.*

ALGEBRA AND EQUATIONS AND INEQUALITIES [24]

QUESTION 1

Answers on p. A1

1.1 Solve for x:

- 1.1.1 $x(x-7) = 0$ (2)
- 1.1.2 $x^2 - 6x + 2 = 0$ *(correct to TWO decimal places)* (3)
- 1.1.3 $\sqrt{x-1} + 1 = x$ (5)
- 1.1.4 $3^{x+3} - 3^{x+2} = 486$ (4)

1.2 Given: $f(x) = x^2 + 3x - 4$

- 1.2.1 Solve for x if $f(x) = 0$ (2)
- 1.2.2 Solve for x if $f(x) < 0$ (2)
- 1.2.3 Determine the values of x for which $f'(x) \geq 0$ (2)

1.3 Solve for x and y: $x = 2y$ and $x^2 - 5xy = -24$ (4) **[24]**

PATTERNS & SEQUENCES [26]

QUESTION 2

Answers on p. A1

Given the finite arithmetic sequence:

5 ; 1 ; –3 ; . . . . . . ; –83 ; –87

2.1 Write down the fourth term (T_4) of the sequence. (1)

2.2 Calculate the number of terms in the sequence. (3)

2.3 Calculate the sum of all the negative numbers in the sequence. (3)

2.4* Consider the sequence:

5 ; 1 ; –3 ; . . . . . . ; –83 ; –87 ; . . . . . . ; –4 187

Determine the number of terms in this sequence that will be exactly divisible by 5. (4) **[11]**

QUESTION 3

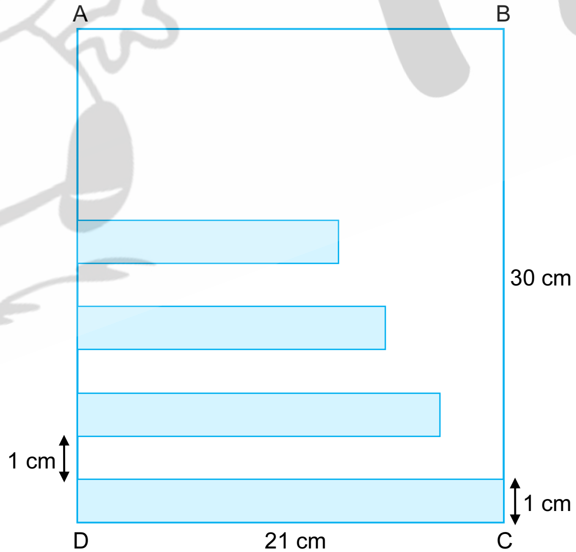
Answers on p. A2

3.1* The first four terms of a quadratic number pattern are -1 ; x ; 3 ; $x+8$

3.1.1 Calculate the value(s) of x. (4)

3.1.2 If $x = 0$, determine the position of the first term in the quadratic number pattern for which the sum of the first n first differences will be greater than 250. (4)

3.2* Rectangles of width 1 cm are drawn from the edge of a sheet of paper that is 30 cm long such that there is a 1 cm gap between one rectangle and the next.
The length of the first rectangle is 21 cm and the length of each successive rectangle is 85% of the length of the previous rectangle until there are rectangles drawn along the entire length of AD. Each rectangle is shaded.



3.2.1 Calculate the length of the 10th rectangle. (3)

3.2.2 Calculate the percentage of the paper that is shaded. (4) **[15]**

FUNCTIONS & GRAPHS [35]

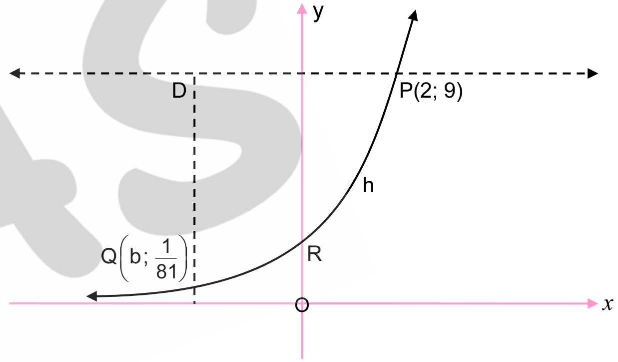
QUESTION 4

Answers on p. A2

Sketched below is the graph of $h(x) = a^x$, $a > 0$.

R is the y-intercept of h.

The points P(2; 9) and $Q\left(b; \frac{1}{81}\right)$ lie on h.



4.1 Write down the equation of the asymptote of h. (1)

4.2 Write down the coordinates of R. (1)

4.3 Calculate the value of a. (2)

4.4 D is a point such that DQ || y-axis and DP || x-axis. Calculate the length of DP. (4)

4.5* Determine the values of k for which the equation $h(x+2) + k = 0$ will have a root that is less than -6. (3)

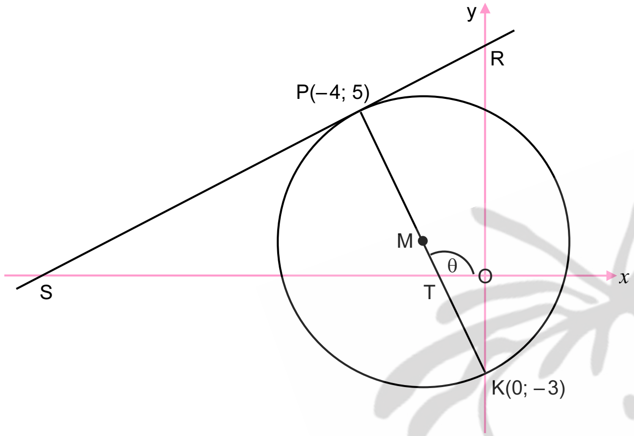
[11]

Finding some challenges along the way? Use the memo wisely and make sure you process and understand every step of the solution.

QUESTION 4

Answers on p. A17

In the diagram, P(–4; 5) and K(0; –3) are the end points of the diameter of a circle with centre M. S and R are respectively the x- and y-intercepts of the tangent to the circle at P.
θ is the inclination of PK with the positive direction of the x-axis.



4.1 Determine:

4.1.1 The gradient of SR (4)

4.1.2 The equation of SR in the form $y = mx + c$ (2)

4.1.3 The equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$ (4)

4.1.4 The size of $P\hat{K}R$ (3)

4.1.5 The equation of the tangent to the circle at K in the form $y = mx + c$ (2)

4.2* Determine the values of t such that the line $y = \frac{1}{2}x + t$ cuts the circle at two different points. (3)

4.3* Calculate the area of ΔSMK. (5)

[23]

See pp. xi & xii for Analytical Geometry Toolkit

▶ TRIGONOMETRY [41]

Trig Summary on p. vii

QUESTION 5

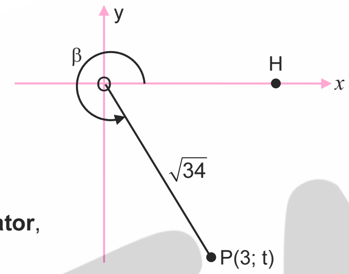
Answers on p. A18

5.1 Given: $\dfrac{\sin(A - 360°) \cdot \cos(90° + A)}{\cos(90° - A) \cdot \tan(-A)}$

Simplify the expression to a single trigonometric ratio. (6)

5.2 In the diagram, P(3; t) is a point in the Cartesian plane.

OP = $\sqrt{34}$ and reflex $H\hat{O}P = \beta$.



Without using a calculator, determine the value of:

5.2.1 t 5.2.2 tan β 5.2.3 cos 2β (2)(1)(4)

5.3* Prove:

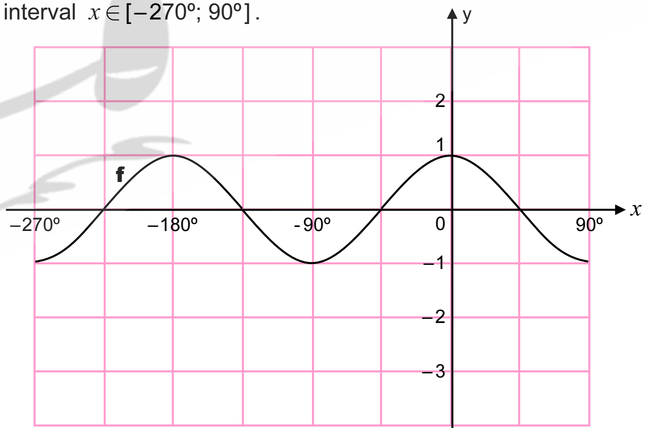
5.3.1 $\sin(A + B) - \sin(A - B) = 2\cos A \cdot \sin B$ (2)

5.3.2 and **hence, without using a calculator**, that $\sin 77° - \sin 43° = \sin 17°$ (4) **[19]**

QUESTION 6

Answers on p. A19

In the diagram, the graph of $f(x) = \cos 2x$ is drawn for the interval $x \in [-270°; 90°]$.



6.1 Draw the graph of $g(x) = 2\sin x - 1$ for the interval $x \in [-270°; 90°]$ on the grid. Show ALL the intercepts with the axes, as well as the turning points. (4)

6.2* Let A be a point of intersection of the graphs of **f** and **g**.

Show that the x-coordinate of A satisfies the equation $\sin x = \dfrac{-1 + \sqrt{5}}{2}$. (4)

6.3* Hence, calculate the coordinates of the points of intersection of graphs of **f** and **g** for the interval $x \in [-270°; 90°]$. (4)

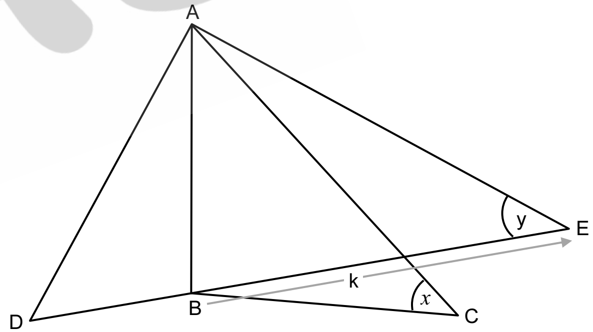
[12]

QUESTION 7

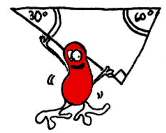
Answers on p. A20

AB represents a vertical netball pole. Two players are positioned on either side of the netball pole at points D and E such that D, B and E are on the same straight line. A third player is positioned at C. The points B, C, D and E are in the same horizontal plane.

The angles of elevation from C to A and from E to A are x and y respectively. The distance from B to E is k.



7.1 Write down the size of $A\hat{B}C$. (1)



7.2* Show that $AC = \dfrac{k \cdot \tan y}{\sin x}$ (4)

7.3* If it is further given that $D\hat{A}C = 2x$ and AD = AC, show that the distance DC between the players at D and C is 2k tan y. (5)

[10]

Consult the Topic Guide on page 2 for more examples on a particular topic.

2.3 All the terms from -3 onwards are negative

$\therefore$ There are 22 terms, i.e. $n = 22$

$\mathbf{a} = -3$; $\mathbf{d} = -4$; $\mathbf{S_{22}}$? ; $\mathbf{T_{22}} = -87$

$\mathbf{S_n = \frac{n}{2}(a + T_n)}$ ➡ $S_{22} = \frac{22}{2}[-3 + (-87)]$

$= 11(-90)$

$= \mathbf{-990}$ ✓

Always preferable to use this formula when you have the value of T_n.

OR: $\mathbf{S_n = \frac{n}{2}[2a + (n-1)d]}$

➡ $S_{22} = \frac{22}{2}[2(-3) + (22-1)(-4)]$

$= 11[-6 - 84]$

$= 11[-90]$

$= \mathbf{-990}$ ✓

2.4 Now, $\mathbf{a} = 5$ & $\mathbf{d} = -4$, but $\mathbf{T_n} = -4\,187$, $\mathbf{n}$?

$\mathbf{T_n = -4n + 9}$ ➡ $-4\,187 = -4n + 9$

$\therefore -4\,196 = -4n$

$\therefore n = 1\,049$

OR: $\mathbf{T_n = a + (n-1)d}$ ➡ $-4\,187$

$= 5 + (n-1)(-4)$, etc.

Terms divisible by 5:

5 ; 1 ; -3 ; -7 ; -11 ; -15 ; -19 ; -23 ; -27 ; -31 ; -35 ; . . .

The first term of each group of 5 terms is divisible by 5.

The number of groups of 5 terms $= \frac{1\,049}{5} = 209$ rem 4.

$\therefore$ There are 209 complete groups and the first 4 terms of the next group (which will start with a multiple of 5).

$\therefore$ **210 terms** ✓

3.1

$-1 \qquad x \qquad 3 \qquad x + 8$

1st diff.: $x + 1 \qquad 3 - x \qquad x + 5$

2nd diff.: $(3 - x) - (x + 1) \qquad (x + 5) - (3 - x)$

$2 - 2x \qquad 2 + 2x$

3.1.1 $2 - 2x = 2 + 2x$ . . . *quadratic sequence has equal 2nd differences*

$\therefore -4x = 0$

$\therefore \mathbf{x = 0}$ ✓

3.1.2 $-1 \quad 0 \quad 3 \quad 8$

$1 \quad 3 \quad 5$

$2 \quad 2$

The 1st differences: 1 ; 3 ; 5 ; . . .

$\therefore \mathbf{T_n = 2n - 1}$

& $\mathbf{S_n = \frac{n}{2}(a + T_n)}$

$= \frac{n}{2}(1 + 2n - 1)$

$= \frac{n}{2}(2n)$

$= n^2$

$S_n = 250 \Rightarrow n^2 = 250$

$\therefore n = 15{,}81...$ . . . $n > 0$

$\therefore S_n > 250 \Rightarrow n = 16$

The 16th term (of the 1st differences) is the 1st difference between the 16th and 17th term of the original number pattern

$\therefore$ **The required position is the 17th term** ✓

Note: The positions of the first differences are 1 behind the positions of the original number pattern.

3.2.1 The **lengths** of the rectangles are:

21 ; $21 \times 0{,}85$; $21 \times (0{,}85)^2$; . . .

$\therefore T_{10} = 21 \times 0{,}85^9$ . . . $\mathbf{T_n = ar^{n-1}}$

$\simeq$ **4,86 cm** ✓

3.2.2 The length of the sheet is 30 cm

$\therefore$ 15 rectangles can be drawn

The sum of the **areas** of the 15 rectangles:

S_{15}

$= 21 \times 1 + 21 \times 0{,}85 \times 1 + 21 \times (0{,}85)^2 \times 1 + \ldots$ . . . *15 terms*

$= \frac{21[1 - (0{,}85)^{15}]}{1 - 0{,}85}$ . . . $\mathbf{S_n = \frac{a(1 - r^n)}{1 - r}}$

$= 127{,}77\ \text{cm}^2$

$\therefore$ The % that is shaded $= \frac{127{,}77}{21 \times 30} = 0{,}20281...$

$\simeq$ **20,28%** ✓

▶ FUNCTIONS AND GRAPHS [35]

4.1 $\mathbf{y = 0}$ ✓ . . . *The asymptote is the x-axis!*

& The **eqn.** *of the x-axis is* $\mathbf{y = 0}$

4.2 **R(0; 1)** ✓ . . . *At R,* $x = 0$ & $y = a^0 = 1$

4.3 Pt P(2; 9) on graph $y = a^x$

Substitute: $\therefore 9 = a^2$

$\therefore \mathbf{a = 3}$ ✓ . . . $a \geq 0$ *in* $y = a^x$

4.4 Pt $Q\left(b; \frac{1}{81}\right)$ on graph $y = 3^x$ . . . *a = 3 in 4.3*

$\therefore \frac{1}{81} = 3^b$

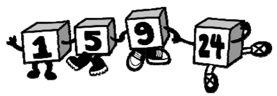
$\therefore 3^b = 3^{-4}$ . . . $\frac{1}{81} = \frac{1}{3^4} = 3^{-4}$

$\therefore b = -4$

$\therefore x_D = -4$ . . . $x_D = x_Q$

$\therefore$ Length of DP $= 4 + 2$ . . . *or,* $2 - (-4)$

$=$ **6 units** ✓



8.1.5 **In ΔONR:**

ON = OR ... *equal radii*

$\therefore\ O\hat{N}R = \hat{R}$

$= \frac{1}{2}(180^\circ - 66^\circ)$ ... *sum of $\angle^s$ of Δ*

$= 57^\circ$

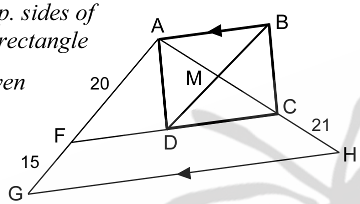
$\therefore\ \hat{N}_2 = 57^\circ - 24^\circ$

$= \mathbf{33^\circ}$

8.2.1 FC || AB ... *opp. sides of a rectangle*

& AB || GH ... *given*

$\therefore$ **FC || GH** ◄



8.2.2 **In ΔAGH:**

$\frac{\mathbf{AC}}{\mathrm{CH}} = \frac{\mathrm{AF}}{\mathrm{FG}}$ ... *prop. theorem; FC || GH*

$\therefore\ \frac{\mathrm{AC}}{21} = \frac{20}{15}$

$\therefore$ AC $= \frac{21 \times 20}{15}$ = 28 units

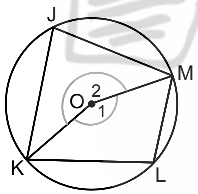
$\therefore$ DB (= AC) = 28 units ... *Diagonals of a rect. are =*

$\therefore$ DM $= \frac{1}{2}(28)$ ... *Diagonals of a $||^m$ (and $\therefore$ a rectangle) bisect one another*

= **14 units** ◄

9.1 Theorem

RTP: $\hat{J} + \hat{L} = 180^\circ$



Construction:

Draw radii OK and OM

Proof:

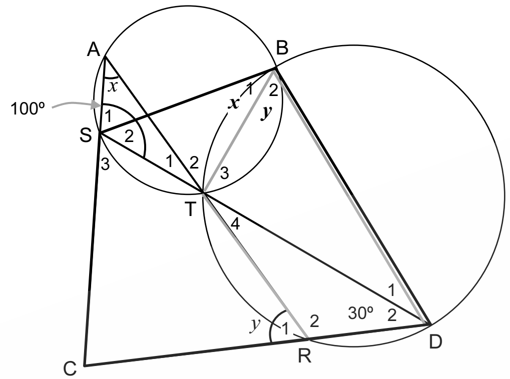
$\hat{J}_1 = \frac{1}{2}\hat{O}_1$ and $\hat{L} = \frac{1}{2}\hat{O}_2$ ... *$\angle$ at centre = 2 × $\angle$ at circumference*

$\therefore\ \hat{J}_1 + \hat{L} = \frac{1}{2}(\hat{O}_1 + \hat{O}_2)$

But $\hat{O}_1 + \hat{O}_2 = 360^\circ$ ... *$\angle^s$ about point O*

$\therefore\ \mathbf{\hat{J}_1 + \hat{L} = 180^\circ}$ ◄

9.2



9.2.1 (a) $\mathbf{\hat{B}_1 = x}$ ◄ ... *$\angle^s$ in the same segment; chord ST subtends*

(b) $\mathbf{\hat{B}_2 = y}$ ◄ ... *exterior $\angle$ of cyclic quadrilateral*

9.2.2 Using 9.2.1 (a) & (b): $S\hat{B}D = x + y$

& **In ΔACR:** $\hat{C} = 180^\circ - (x + y)$ ... *sum of $\angle^s$ of Δ*

$\therefore$ **In quadrilateral SCDB:**

$S\hat{B}D$ and $\hat{C}$ are supp. $\angle^s$

Here, we use the CONVERSE of the 'opp. $\angle^s$ of c.q.' thm.

$\therefore$ **SCDB is a cyclic quad.** ◄

... *CONVERSE of 'opp. $\angle^s$ of cyclic quad.' theorem*

9.2.3

We need to prove: $x + y \neq 90^\circ$

$\hat{C} = A\hat{S}T - \hat{D}_2$ ... *ext. $\angle$ of ΔSCD*

$= 100^\circ - 30^\circ$

$= 70^\circ$

$\therefore\ S\hat{B}D\ (= x + y) = 110^\circ$ ... *opp. $\angle^s$ of c.q. SCDB*

$\therefore\ x + y \neq 90^\circ$

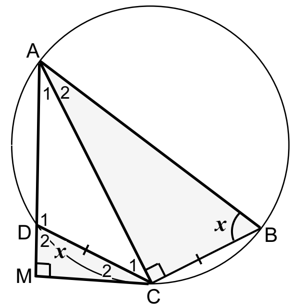
$\therefore$ **SD is not a diameter of circle BDS** ◄

... *CONVERSE of '$\angle$ in semi-$\odot$' thm.*

10.1.1 We need to prove that

$\hat{C}_2 = \hat{A}_1$

We will use the CONVERSE of the tan chord theorem



$\hat{D}_2 = \hat{B}$ ... *exterior $\angle$ of cyclic quad.*

$= x$

$\therefore$ In ΔDMC:

$\hat{C}_2 = 90^\circ - x$ ... *sum of $\angle^s$ of Δ*

But $\hat{A}_1 = \hat{A}_2$ ... *equal chords subtend equal angles*

& In ΔACB: $\hat{A}_2 = 90^\circ - x$ ... *sum of $\angle^s$ of Δ*

$\therefore\ \hat{A}_1 = 90^\circ - x$

$\therefore\ \hat{C}_2 = \hat{A}_1$

$\therefore$ **MC is a tangent to the circle at C** ◄

... *CONVERSE of tan chord theorem*

10.1.2 In Δ^s ACB and CMD

(1) $A\hat{C}B = \hat{M}\ (= 90^\circ)$ ... *given*

(2) $\hat{A}_2 = \hat{C}_2$ ... *both = $90^\circ - x$ in 10.1.1*

$\therefore$ **ΔACB ||| ΔCMD** ◄ ... *$\angle\angle\angle$*

10.2.1 $\frac{\mathrm{CM}}{\mathrm{DC}} = \frac{\mathrm{AC}}{\mathrm{AB}}$ ... ❶ ... *similar Δ^s in 10.1.2*

But, in Δ^s CMD and AMC

(1) $\hat{M}$ (= 90°) is common

(2) $\hat{C}_2 = \hat{A}_1$ ... *proved in 10.1.1*

$\therefore$ ΔCMD ||| ΔAMC ... *$\angle\angle\angle$*

$\therefore\ \frac{\mathrm{CM}}{\mathrm{DC}} = \frac{\mathrm{AM}}{\mathrm{AC}}$ ... ❷ ... *||| Δ^s*

$\therefore\ \frac{\mathrm{CM}}{\mathrm{DC}} \times \frac{\mathrm{CM}}{\mathrm{DC}} = \frac{\mathrm{AC}}{\mathrm{AB}} \times \frac{\mathrm{AM}}{\mathrm{AC}}$ ... *see ❶ and ❷*

$\therefore\ \frac{\mathbf{CM^2}}{\mathbf{DC^2}} = \frac{\mathbf{AM}}{\mathbf{AB}}$ ◄

10.2.2 $\frac{\mathrm{AM}}{\mathrm{AB}} = \frac{\mathrm{CM}^2}{\mathrm{DC}^2}$ ... *proved in 10.2.1*

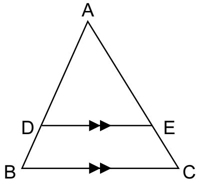
But, in ΔDMC: $\frac{\mathrm{CM}}{\mathrm{DC}} = \sin x$

$\therefore\ \frac{\mathbf{AM}}{\mathbf{AB}} = \mathbf{\sin^2 x}$ ◄

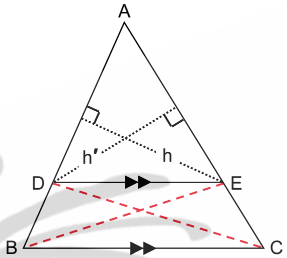
▶ The Proportion Theorem

6 A line parallel to one side of a triangle divides the other two sides proportionally.

i.e. DE || BC ➡ $\frac{AD}{DB} = \frac{AE}{EC}$



Given: $\triangle ABC$ with DE || BC,
D & E on AB & AC respectively.



To prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: Join DC & BE

Proof: $\frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle DBE} = \frac{\frac{1}{2}AD.h}{\frac{1}{2}DB.h} = \frac{AD}{DB}$

h is the height of $\triangle^s$ ADE and DBE

h′ is the height of $\triangle^s$ ADE and EDC

Similarly: $\frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle EDC} = \frac{AE}{EC}$ $\left(\frac{\frac{1}{2}AE.h'}{\frac{1}{2}EC.h'}\right)$

But: $\triangle DBE = \triangle EDC$, in area . . . *on the same base DE ; between || lines, DE & BC*

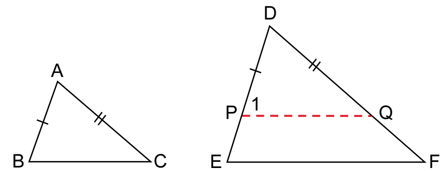
and: $\triangle ADE$ is common

$\therefore \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle DBE} = \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle EDC}$

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$ ◀

▶ The Similar $\triangle^s$ Theorem

7 If two triangles are equiangular, then their sides are proportional and, therefore, they are similar.



Given: $\triangle ABC$ & $\triangle DEF$ with $\hat{A} = \hat{D}$ $\hat{B} = \hat{E}$ & $\hat{C} = \hat{F}$

To prove: $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Construction: Mark P & Q on DE & DF such that DP = AB & DQ = AC

Proof: In $\triangle^s$ DPQ & ABC

(1) DP = AB . . . *construction*
(2) DQ = AC . . . *construction*
(3) $\hat{D} = \hat{A}$ . . . *given*

$\therefore \triangle DPQ \equiv \triangle ABC$ . . . *S∠S*

1 congruency

$\therefore \hat{P}_1 = \hat{B}$
$= \hat{E}$ . . . *given*

2 corresponding $\angle^s$

The focal point

$\therefore$ PQ || EF . . . *corresponding $\angle^s$ equal*

3 parallel lines

$\therefore \frac{DP}{DE} = \frac{DQ}{DF}$ . . . *proportion theorem ; PQ || EF*

But DP = AB and
DQ = AC . . . *construction*

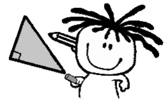
$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

4 proportions

Similarly, by marking S and T on DE and EF such that SE = AB and ET = BC, it can be proved that: $\frac{AB}{DE} = \frac{BC}{EF}$

$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ ◀

$\therefore \triangle ABC$ and $\triangle DEF$ are similar.



Similar $\triangle^s$

$\triangle^s$ are similar if: **A:** they are equiangular, and
B: their sides are proportional

In this proof, we show that: **A** ➡ **B**
i.e. Both conditions, **A** and **B**, apply
$\therefore$ The $\triangle^s$ are similar

The converse statement says: **B** ➡ **A**
i.e. Both conditions, **A** and **B**, apply
$\therefore$ *The $\triangle^s$ are similar*

Compound Angle Formulae

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Sign stays the same
sine & cosine of A and B mixed

3. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Sign changes
cosine of A and B first, then sine of A & B

We will prove formula no. 4 (see above) and then derive the other 3 from it.

Double Angle Formulae

5. $\sin 2A = 2 \sin A \cos A$ . . . This formula will be derived from the formula no. 1.

6. $\cos 2A = \cos^2 A - \sin^2 A$ . . . This formula will be derived from the formula no. 3.

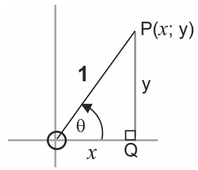
or $\cos 2A = 1 - 2\sin^2 A$

or $\cos 2A = 2\cos^2 A - 1$

Proof of the Formula:

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

First, an important concept!



NOTE: **If OP = 1 unit!**

then: $\frac{x}{1} = \cos\theta$ and $\frac{y}{1} = \sin\theta$

i.e. $x = \cos\theta$ and $y = \sin\theta$

i.e. **P is the point $(\cos\theta;\ \sin\theta)$**

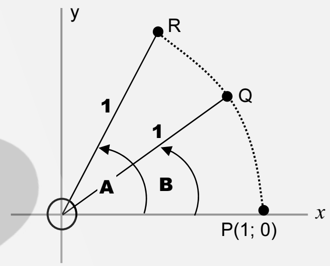
In the sketch alongside, $\hat{A}$ and $\hat{B}$ have been placed in standard position.

$R\hat{O}Q = \hat{A} - \hat{B}$.

The coordinates of the points **R** and **Q**, both **1 unit** from the origin, are:

R(cos A ; sin A) & **Q(cos B ; sin B)**

. . . See NOTE above



▶ **Determine 2 expressions for RQ²**

RQ^2 $= 1^2 + 1^2 - 2(1)(1)\cos(A - B)$ . . . **COSINE RULE**

$= 2 - 2\cos(A - B)$ . . . ❶

& **RQ^2** $= (\cos A - \cos B)^2 + (\sin A - \sin B)^2$ . . . **DISTANCE FORMULA**

$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$

$= 2 - 2\cos A \cos B - 2\sin A \sin B$ ◀ . . . ❷ . . . $\sin^2\theta + \cos^2\theta = 1$

▶ **Equate the two expressions for RQ² above:**

❶ = ❷ $\therefore\ 2 - 2\cos(A - B) = 2 - 2\cos A \cos B - 2\sin A \sin B$

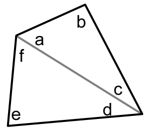
▶ **Subtract 2:** $\therefore\ -2\cos(A - B) = -2\cos A \cos B - 2\sin A \sin B$

▶ **Divide by −2** $\left(\text{or } \times \text{ by } -\frac{1}{2}\right)$: $\therefore$ **$\cos(A - B) = \cos A \cos B + \sin A \sin B$** ◀

QUADRILATERALS – definitions, areas & properties

All you need to know

'Any' Quadrilateral



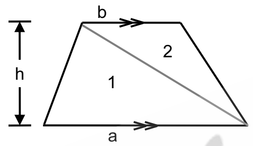
Sum of the $\angle^s$ of *any quadrilateral* = 360°

Sum of the interior angles
= (a + b + c) + (d + e + f)
= 2 × 180° . . . (2 Δ^s)
= 360°

*The arrows indicate various '**pathways**' from 'any' quadrilateral to the square (the 'ultimate quadrilateral'). These pathways, which combine logic and fact, are essential to use when proving specific types of quadrilaterals.*

See how the properties accumulate as we move from left to right, i.e. the first quad. has no special properties and each successive quadrilateral has all preceding properties.

A Trapezium

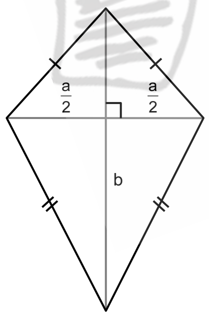


DEFINITION:
Quadrilateral with **1 pair of opposite sides ||**

Area = $\Delta 1 + \Delta 2$
$= \frac{1}{2}ah + \frac{1}{2}bh$
$= \mathbf{\frac{1}{2}(a+b).h}$

'Half the sum of the || sides × the distance between them.'

A Kite



DEFINITION:
Quadrilateral with **2 pairs of adjacent sides equal**

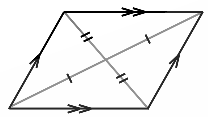
Given diagonals a and b

Area $= 2\Delta^s = 2\left(\frac{1}{2}b.\frac{a}{2}\right) = \frac{ab}{2}$

'Half the product of the diagonals'

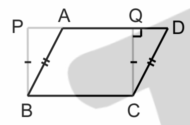
THE DIAGONALS
- *cut perpendicularly*
- ***ONE DIAGONAL** bisects the other diagonal, the opposite angles and the area of the kite*

A Parallelogram



DEFINITION:
Quadrilateral with **2 pairs opposite sides ||**

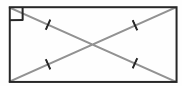
Area = base × height



$||^m$ ABCD = ABCQ + ΔQCD
rect. PBCQ = ABCQ + ΔPBA
where ΔQCD ≡ ΔPBA . . . *RHS/90°HS*
∴ $||^m$ ABCD = rect. PBCQ (in area)
= BC × QC

Properties:
2 pairs opposite sides equal
2 pairs opposite angles equal
*& **DIAGONALS BISECT ONE ANOTHER***

A Rectangle

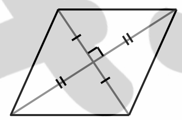


DEFINITION:
A $||^m$ with one right ∠

Area = ℓ × b

***DIAGONALS** are EQUAL*

A Rhombus

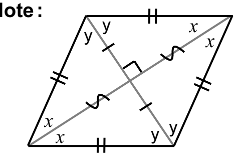


DEFINITION:
A $||^m$ with one pair of adjacent sides equal

Area
= $\mathbf{\frac{1}{2}}$ **product of diagonals** (as for a kite)
or
= **base × height** (as for a parallelogram)

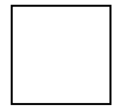
THE DIAGONALS
- *bisect one another PERPENDICULARLY*
- *bisect the angles of the rhombus*
- *bisect the area of the rhombus*

Note:



2x + 2y = 180° . . . $\angle^s$ *of* Δ *or co-int.* $\angle^s$*; || lines*
➡ x + y = 90°

The Square

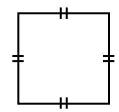


the 'ultimate' quadrilateral!

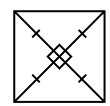
Area = s²

Properties:
It's all been said 'before'!
Since a square is a rectangle, a rhombus, a parallelogram, a kite, . . . ALL the properties of these quadrilaterals apply.

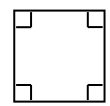
sides



diagonals



angles

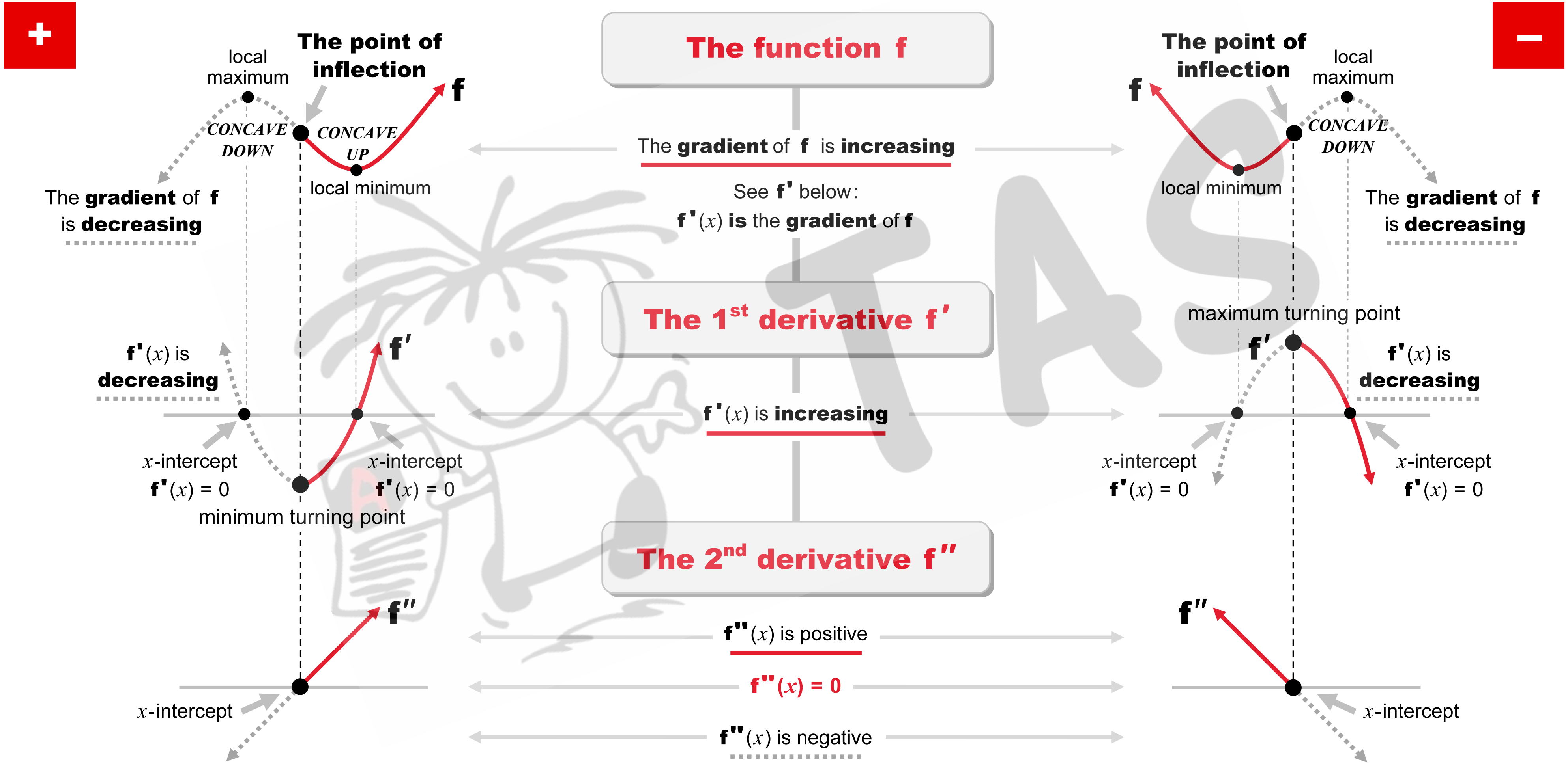


Quadrilaterals play a prominent role in both Euclidean & Analytical Geometry right through to Grade 12!

CONCAVITY & THE POINT OF INFLECTION

The **Concavity** of cubic graphs: **Concave up** or **Concave down**, changes at the point of inflection:

As x increases (i.e. from left to right) . . .



Note: For cubic graphs with 2 stationary points, the coordinates of the point of inflection are the averages of the x- and y-coordinates of the stationary points.

$f''(x) = 0$ at the point of inflection of f

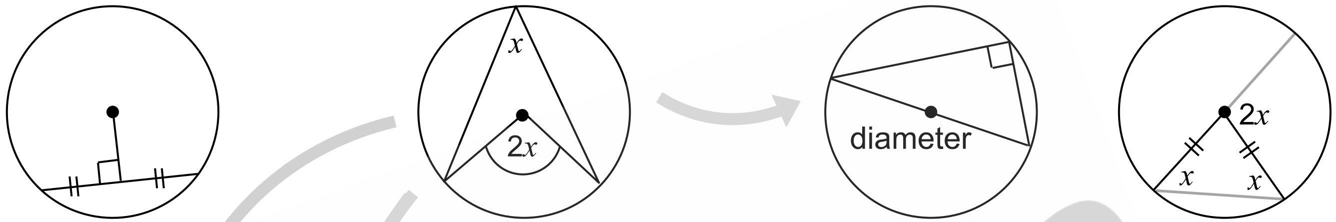
$f''(x) < 0$ where f is concave down

$f''(x) > 0$ where f is concave up

GROUPING OF CIRCLE GEOMETRY THEOREMS

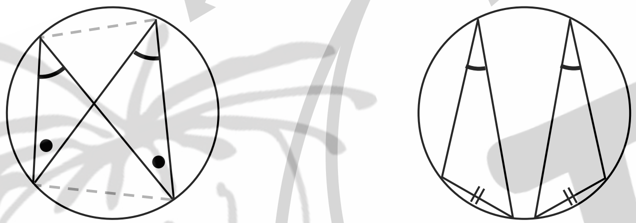
The grey arrows indicate how various theorems are used to prove subsequent ones

I The 'Centre' group



Equal radii!

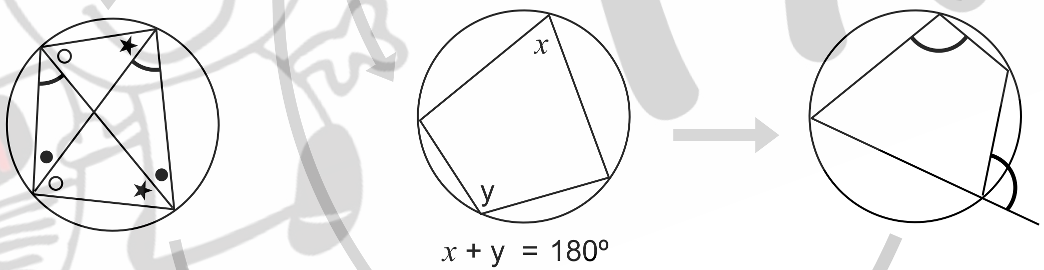
II The 'No Centre' group



Equal chords!

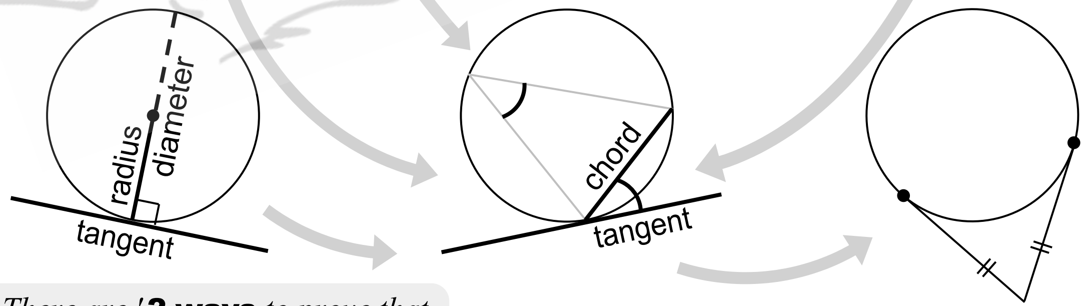
Equal chords subtend equal angles and, vice versa, equal angles are subtended by equal chords.

III The 'Cyclic Quad.' group



*There are '**3 ways** to prove that a quad. is a cyclic quad.'.*

IV The 'Tangent' group



Equal tangents!

*There are '**2 ways** to prove that a line is a tangent to a ⊙'.*