INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2023

FURTHER STUDIES MATHEMATICS (STANDARD): PAPER I

MARKING GUIDELINES

Time: 2 hours 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

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QUESTION 1

1.1 Solve:

(a) $\ln(2+e^{-x})=2$ Leave your answer in the form $x=\ln(....)$

$e^2=2+e^{-x}$
$e^{-x}=e^2-2$
$-x=\ln(e^2-2)$
$x=-\ln(e^2-2)$
$x=\ln\left(\frac{1}{e^2-2}\right)$

(b) $|2x+3|=3x+4$

$2x+3=3x+4$ or $2x+3=-3x-4$
$x=-1$ or $x=-\frac{7}{5}$
a check reveals $x=-1$ only

1.2 Give, in standard $ax^4+bx^3+cx^2+dx+e=0$ form, a quartic equation which has $x=2+\sqrt{3}$ and $2-i$ as roots. The values of *a, b, c, d* and *e* must be rational.

one quadratic has roots $2+\sqrt{3}$ and $2-\sqrt{3}$
the sum of these roots is 4 and the product is 1 so $x^2-4x+1=0$
the other has roots $2-i$ and $2+i$
the sum of these roots is 4 and the product is 5 so $x^2-4x+5=0$
$(x^2-4x+1)(x^2-4x+5)=0$
$x^4-8x^3+22x^2-24x+5=0$

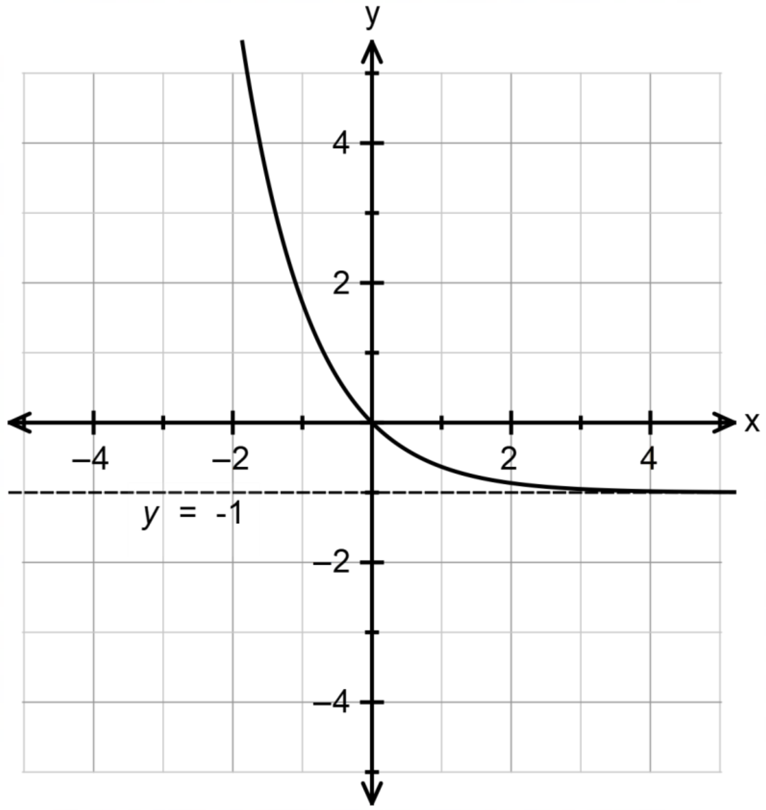
1.3 Determine positive real values of *a* and *b* if:

$(a+bi)(b+i)=(2b+a)i$
$\text{LHS}=ab-b+(b^2+a)i$

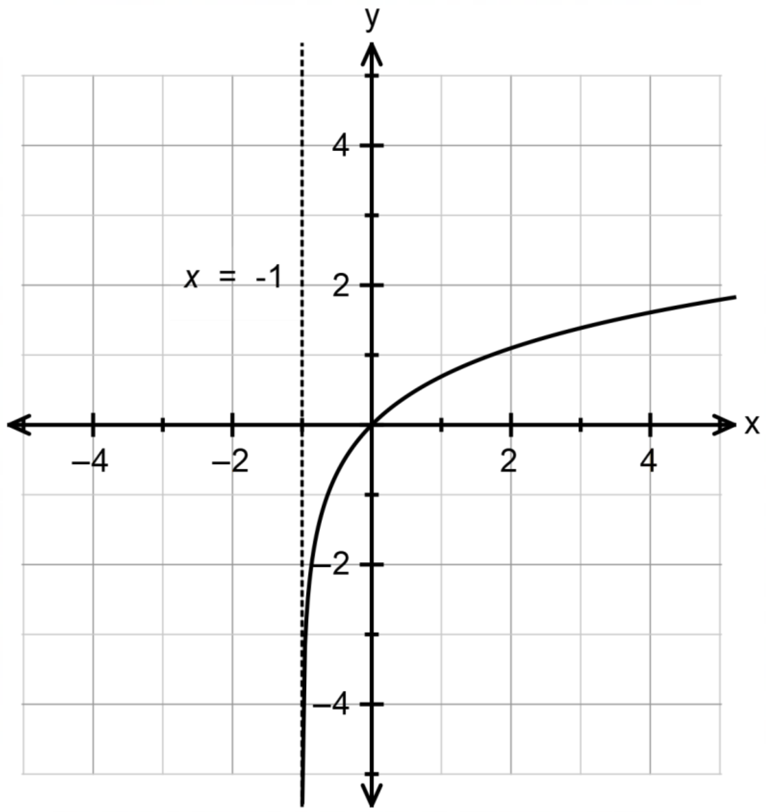
so, $ab-b=0$ and $b^2+a=2b+a$
so, $b(a-1)=0$
since $b\neq 0$, $a=1$
$\therefore b=2$

1.4. Sketch the following functions on the axes provided. You should draw and give the equations of any asymptotes as well as showing any intercepts with the axes.

(a) $y = e^{-x} - 1$



(b) $y = \ln(x+1)$



QUESTION 2

Use mathematical induction to prove that:

$$-1+4-9+16-25+.....(-1)^n n^2 = \frac{(-1)^n n(n+1)}{2} \text{ for } n \in \mathbb{N}$$

if $n=1$ LHS $=-1$ and $RHS=-1$ so it is true for $n=1$

assume true for $n=k$

$$-1+4-9+16-25+.....(-1)^k k^2 = \frac{(-1)^k k(k+1)}{2} (*)$$

now consider $n=k+1$

$$-1+4-9+16-25+.....(-1)^k k^2 + (-1)^{k+1}(k+1)^2 = \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$$

$$= (-1)^k \left[\left(\frac{k(k+1)}{2} - (k+1)^2\right]\right.$$

$$= (-1)^k \left[\left(\frac{k(k+1)}{2} - \frac{2(k+1)^2}{2}\right]\right.$$

$$= (-1)^k \left[\frac{(k+1)(k-2(k+1))}{2}\right]$$

$$= (-1)^k \left[\frac{(k+1)(-k-2)}{2}\right]$$

$$= (-1)^{k+1} \left[\frac{(k+1)(k+2)}{2}\right]$$

but this is just (*) with $n=k+1$

so, we have proved it true for $n=k+1$

$\therefore$ by *PMI* we have proved it for $n \in \mathbb{N}$

QUESTION 3

Determine $\frac{d}{dx}\sqrt{3x}$ by first principles

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \times \frac{\sqrt{3(x+h)} + \sqrt{3x}}{\sqrt{3(x+h)} + \sqrt{3x}}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h) - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{3x + 3h - 3x}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{3h}{h\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{3}{\left(\sqrt{3(x+h)} + \sqrt{3x}\right)}$$

$$f'(x) = \frac{3}{2\sqrt{3x}}$$

QUESTION 4

Consider the function $f(x)=\dfrac{x^2-5x+7}{x-2}$.

(a) Determine, with classification, the equations of any asymptotes.

vertical asymptote: $x=2$

$x^2-5x+7=(x-2)(x-3)+1$

so, $\dfrac{x^2-5x+7}{x-2}=x-3+\dfrac{1}{x-2}$

so, $y=x-3$ is an oblique asymptote

(b) Justify mathematically why the function does not have any x-intercepts.

$\dfrac{x^2-5x+7}{x-2}=0$

$\therefore x^2-5x+7=0$

but $\Delta=(-5)^2-4(1)(7)=-3$

$\therefore$ no real roots

(c) Determine the coordinates of any stationary points.

$f(x)=\dfrac{x^2-5x+7}{x-2}$

$f'(x)=\dfrac{(2x-5)(x-2)-1(x^2-5x+7)}{(x-2)^2}$

$f'(x)=\dfrac{x^2-4x+3}{(x-2)^2}=0$

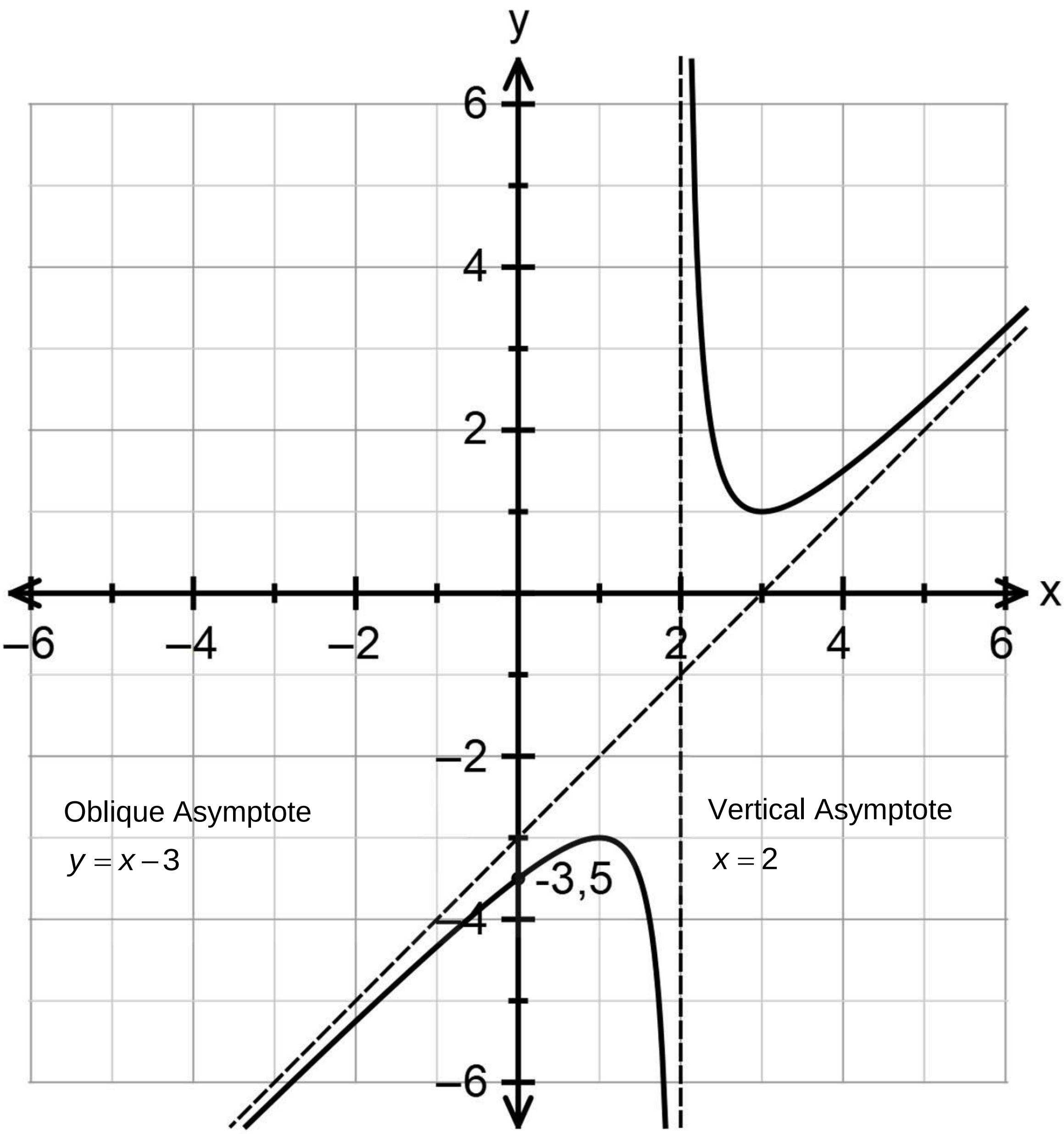
$\therefore x^2-4x+3=0$

$\therefore (x-1)(x-3)=0$

$\therefore x=1 \quad \text{or} \quad 3$

$\therefore (1\,;-3) \quad \text{or} \quad (3\,;1)$

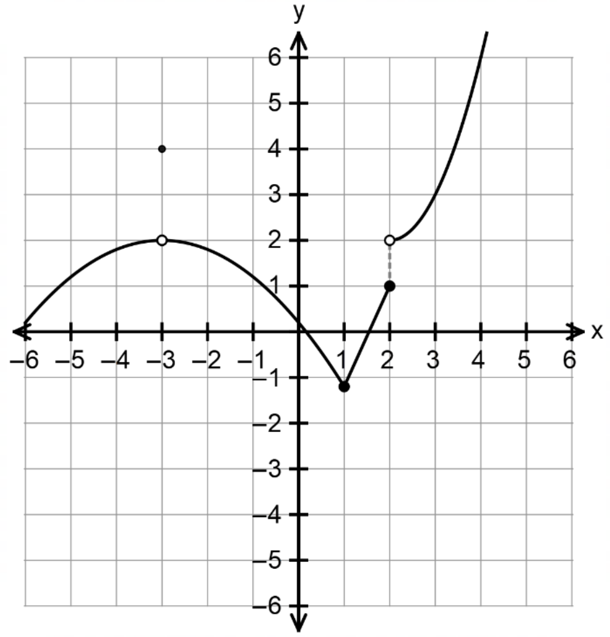
(d) Draw the graph of *f* on the axes provided showing all points of interest. You should draw and label any asymptotes.



QUESTION 5

5.1 On the axes provided draw a **function** g which satisfies the following:

- g is continuous for all values of x except at $x = -3$ and $x = 2$
- $g(-3) = 4$ and $\lim_{x \to -3} g(x)$ exists
- $g(2) = 1$ and $\lim_{x \to 2^-} g(x) = 1$ but there is a jump discontinuity at x = 2
- g is also not differentiable at $x = 1$



5.2 Express the following statements **using mathematical notation**:

(a) The left-hand and right-hand limits of g at a are unequal.

$$\lim_{x \to a^-} g(x) \neq \lim_{x \to a^+} g(x)$$

(b) h is not differentiable at p despite being continuous at p.

$$\lim_{x \to p^-} h'(x) \neq \lim_{x \to p^+} h'(x)$$

5.3 Answer true or false to each of the following statements:

(a) If a function is differentiable at a point, then it is continuous at that point.

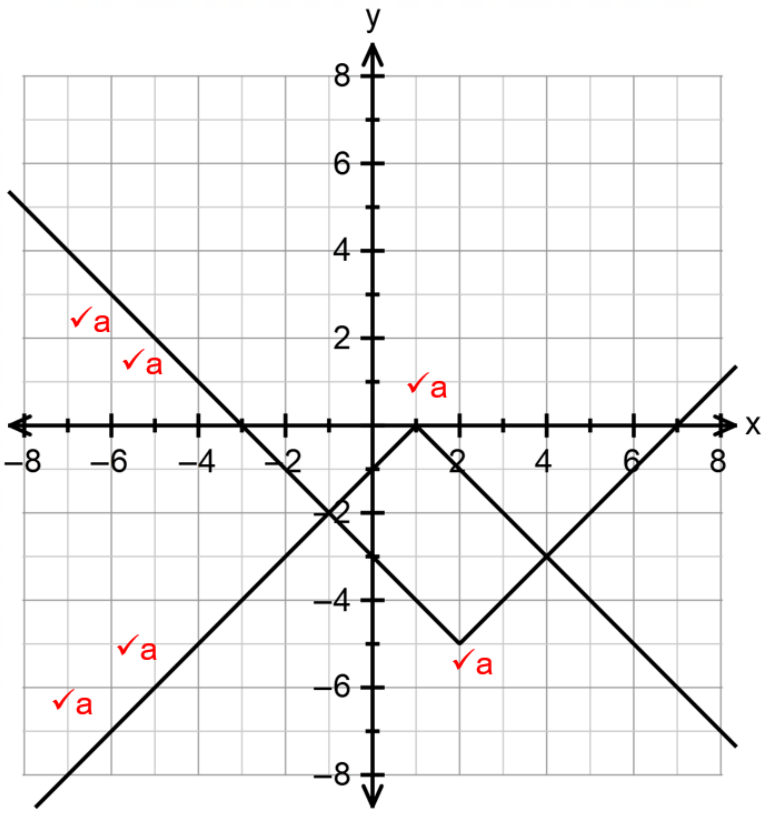
TRUE

(b) If a function is not differentiable at a point, then it is not continuous at that point.

FALSE

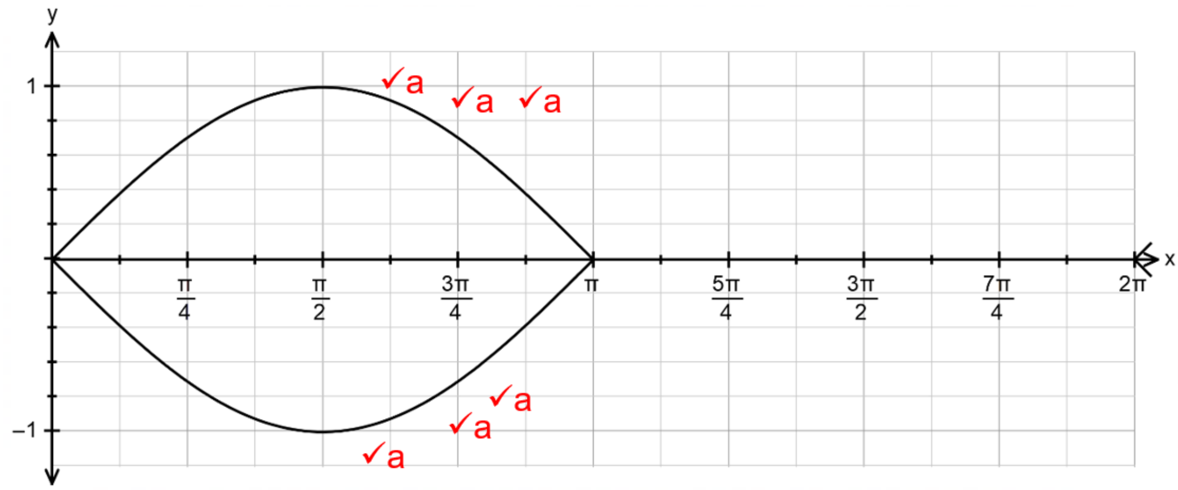
QUESTION 6

6.1 Use the axes below to solve $|x-2|-5 \geq -|x-1|$ sketching the **graphs of two functions**. You must label the graphs you have drawn with their equations.



$x \leq -1$ or $x \geq 4$

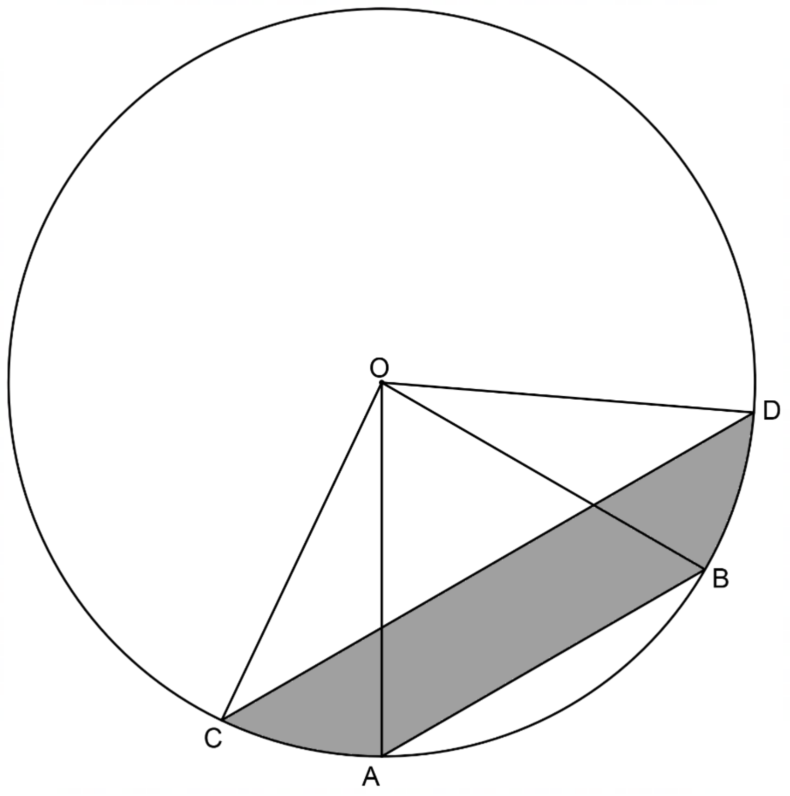
6.2 Draw the graph of $|y| = \sin x$ on the axes provided showing all points of interest.



QUESTION 7

In the diagram below triangle AOB is equilateral with sides of 1 unit.
O is the centre of the circle and $CD = \sqrt{3}$ units.

Determine the shaded area.



$$\cos\left(C\hat{O}D\right) = \left(\frac{1^2 + 1^2 - \left(\sqrt{3}\right)^2}{2(1)(1)}\right) = -\frac{1}{2}$$

$$\therefore C\hat{O}D = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \frac{2\pi}{3}$$

Area sector $OCD = \frac{1}{2}(1^2)\left(\frac{2\pi}{3}\right) = \frac{2\pi}{6}$

from this we must subtract $\Delta OCD = \frac{1}{2}(1)(1)\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}$

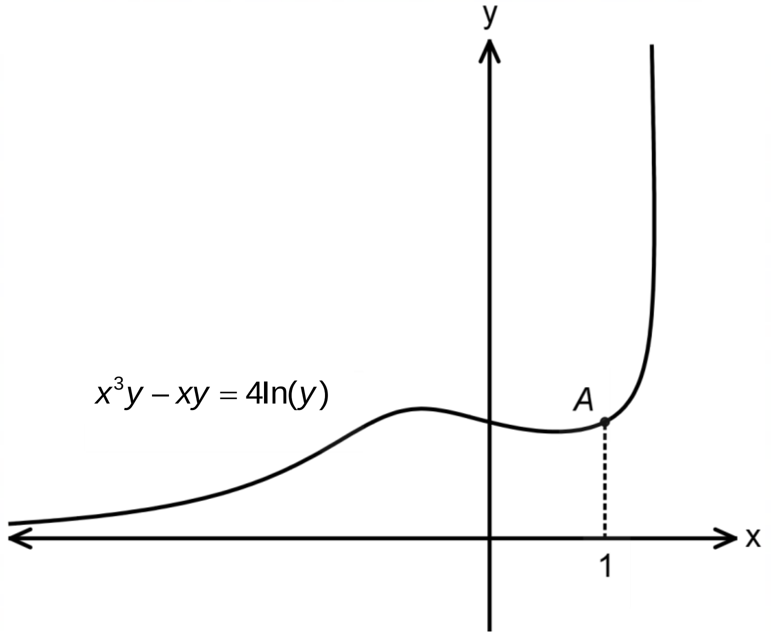
and the segment formed by $AB = \frac{1}{2}\left(1^2\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$

so, shaded area $= \left(\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$

so, shaded area = $\frac{\pi}{6} = 0{,}52$ units^2

QUESTION 8

A portion of the graph of the implicitly defined relationship $x^3y - xy = 4\ln(y)$ is shown below.



(a) Determine the *y*-coordinate of point A showing all working.

$1^3y - 1y = 4\ln(y)$
$0 = 4\ln(y)$
$\ln(y) = 0$
$y = 1$
$\therefore A(1\,;1)$

(b) Find the equation of the tangent to the curve at the point A.

$x^3y - xy = 4\ln(y)$

implicit differentiation yields:

$$3x^2y + x^3\frac{dy}{dx} - y - x\frac{dy}{dx} = \frac{4}{y}\cdot\frac{dy}{dx}$$

$$x^3\frac{dy}{dx} - x\frac{dy}{dx} - \frac{4}{y}\cdot\frac{dy}{dx} = y - 3x^2y$$

$$\frac{dy}{dx}\left(x^3 - x - \frac{4}{y}\right) = y - 3x^2y$$

$$\frac{dy}{dx} = \frac{y - 3x^2y}{x^3 - x - \frac{4}{y}}$$

at $A, \frac{dy}{dx} = \frac{1}{2}$

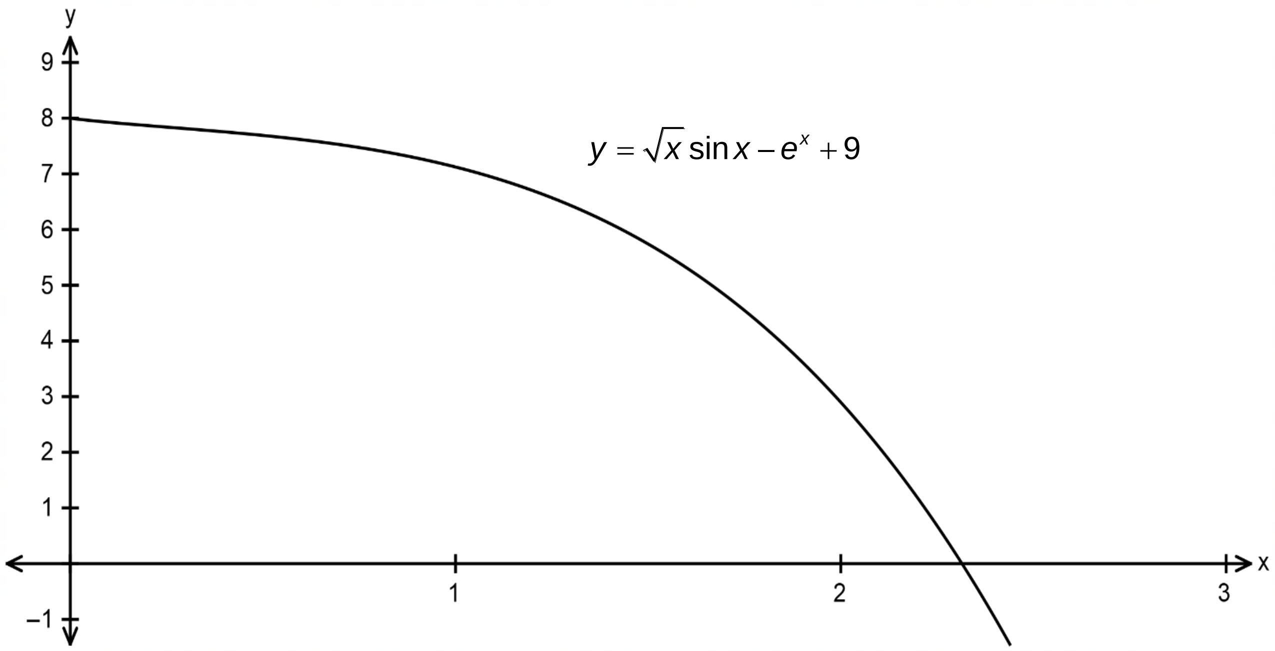
$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

QUESTION 9

The function $f(x) = \sqrt{x}\sin x - e^x + 9$ is shown below.

Use the Newton-Raphson method to find the x-intercept to 5 decimal places using $x_0 = 2$ as an initial guess.



You should show:

- the iterative formula you use.
- x_1 to 5 decimal places.

You do **not** need to write down all your approximations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

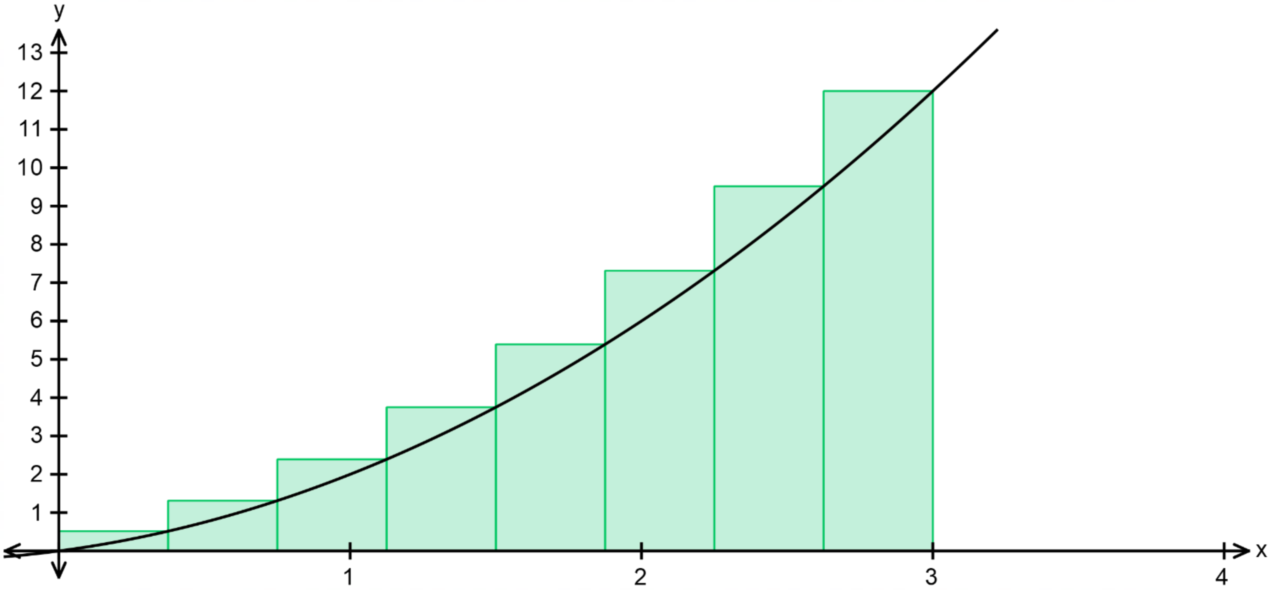
$$x_{n+1} = x_n - \frac{\sqrt{x}_n \sin x_n - e^{x_n} + 9}{\frac{1}{2}x^{-\frac{1}{2}}\sin x_n + \sqrt{x}_n \cos x_n - e^{x_n}}$$

$x_1 = 2{,}37838$

$x = 2{,}31448$

QUESTION 10

Kofi is attempting to work out the area under the curve $y = x^2 + x$ from $x = 0$ to $x = 3$ by partitioning it into rectangles as shown.



He has correctly worked out that when he uses n rectangles the area is given by:

$$A = 13{,}5 + \frac{18}{n} + \frac{27}{6n^2}.$$

He uses his formula and ends up with an error of $13\frac{2}{3}\%$. How many rectangles did he use?

exact answer $= \int_0^3 x^2 + x \, dx = 13{,}5$

$\therefore \frac{A - 13{,}5}{13{,}5} \times 100 = \frac{41}{3}$

$\therefore A = \frac{41}{300} \times 13{,}5 + 13{,}5$

$\therefore A = 15{,}345$

$\therefore 13{,}5 + \frac{18}{n} + \frac{27}{6n^2} = 15{,}345$

$\therefore 81n^2 + 108n + 27 = 92{,}07n^2$

$\therefore 11{,}07n^2 - 108n - 27 = 0$

$\therefore n = 10$

QUESTION 11

Determine the following integrals:

(a) $\int \sin^2 x\, dx$

$\cos 2x = 1 - 2\sin^2 x$

$\therefore 2\sin^2 x = 1 - \cos 2x$

$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$= \int \frac{1}{2} - \frac{1}{2}\cos 2x\ dx$

$= \frac{x}{2} - \frac{\sin 2x}{4} + c$

(b) $\int x\sqrt{x+1}\ dx$

let $u = x+1$ then $x = u-1$

and $du = dx$

$= \int (u-1)u^{\frac{1}{2}}\ du$

$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

Alternatively:

$\int x\sqrt{x+1}\ dx$

by parts

let $f(x) = x$ then $f'(x) = 1$ and let $g'(x) = (x+1)^{\frac{1}{2}}$ then $g(x) = \frac{2}{3}(x+1)^{\frac{3}{2}}$

$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\ dx$

$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\ dx$

$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3} \times \frac{2}{5}(x+1)^{\frac{5}{2}}$

$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c$

(c) $\int \frac{2x+3}{x^2+6x+9}dx$

$$\frac{2x+3}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

$$\therefore 2x+3 = A(x+3)+B = Ax+(3A+B)$$

$$\therefore A = 2 \text{ and } 3(2)+B = 3 \text{ so } B = -3$$

$$= \int \frac{2}{x+3}dx - \int \frac{3}{(x+3)^2}dx$$

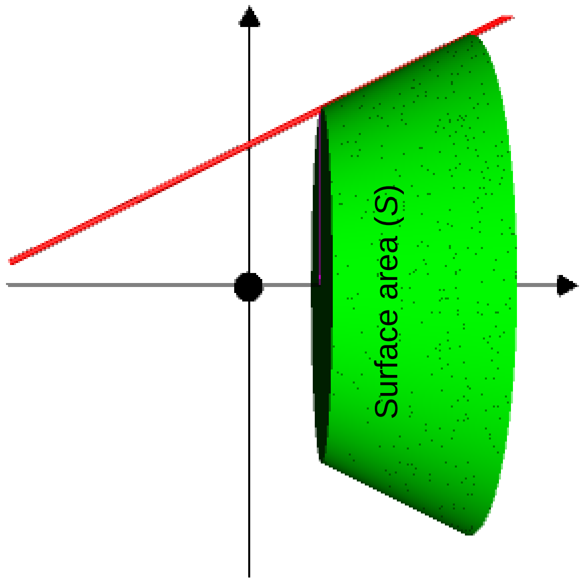
$$= 2\ln|x+3| - 3\int (x+3)^{-2}\,dx$$

$$= 2\ln|x+3| + 3(x+3)^{-1} + c$$

QUESTION 12

Consider the function $y = \frac{x}{2} + 4$.

(a) It is rotated about the x-axis from $x = 2$ to $x = b$ generating a volume of $\frac{436\pi}{3}$ units³.



By setting up and evaluating an integral determine the value of b.

$$\pi\int_2^b \left(\frac{x}{2}+4\right)^2 dx = \frac{436\pi}{3}$$

$$\int_2^b \left(\frac{x}{2}+4\right)^2 dx = \frac{436}{3}$$

$$\int_2^b \frac{x^2}{4}+4x+16\,dx = \frac{436}{3}$$

$$\left[\frac{x^3}{12}+2x^2+16x\right]_2^b = \frac{436}{3}$$

$$\frac{b^3}{12}+2b^2+16b-\left(\frac{2^3}{12}+2(2^2)+16(2)\right) = \frac{436}{3}$$

$$\frac{b^3}{12}+2b^2+16b = \left(\frac{2^3}{12}+2(2^2)+16(2)\right)+\frac{436}{3}$$

$$\frac{b^3}{12}+2b^2+16b = 186$$

$$b^3+24b^2+192b-2\ 232 = 0$$

$$b = 6$$

(b) The surface area (S) generated by rotating f about the x-axis from $x = a$ to $x = b$ is given by the formula:

$$S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Determine the surface area when the function is rotated about the x-axis from $x = 2$ to $x = 6$.

$$S = 2\pi\int_2^6 \left(\frac{x}{2}+4\right)\sqrt{1+(\tfrac{1}{2})^2}\,dx$$

$$S = \sqrt{5}\pi\int_2^6 \left(\frac{x}{2}+4\right)dx$$

$$S = 24\sqrt{5}\pi \text{ units}^2$$

Total: 200 marks

INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2023

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

MARKING GUIDELINES

Time: 1 hour | 100 marks

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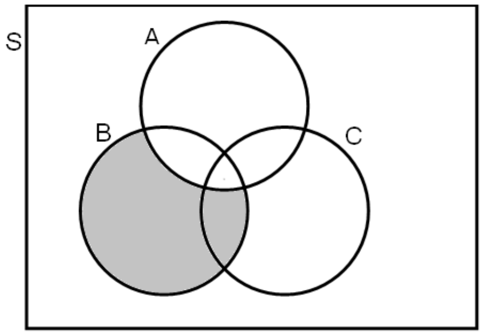
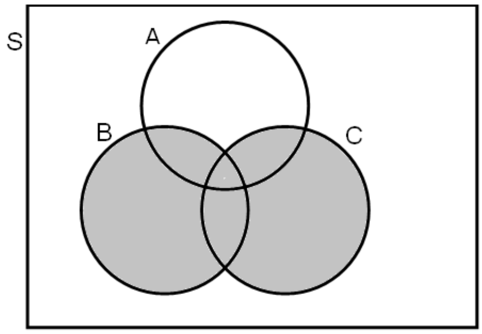
MODULE 2 STATISTICS

QUESTION 1

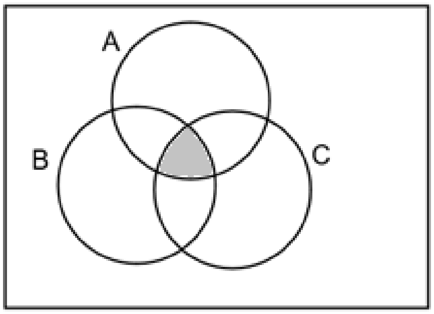
1.1 (a)



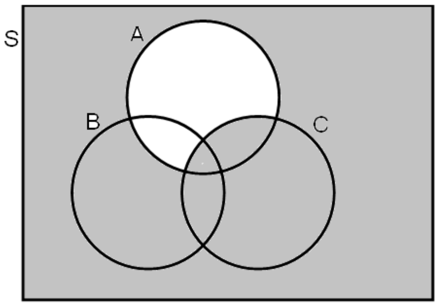
(b)



(c)



(d)



1.2 (a) $P(F \cap L) = \frac{63}{180} = \frac{7}{20}$

(b) $P(F) \times P(L) = P(F \cap L)$

$\frac{108}{180} \times \frac{x}{180} = \frac{7}{20}$

$x = 105$

QUESTION 2

$X \sim N(10;4^2)$

(a) $$P(12 < X < 15) = P\left(\frac{12-10}{4} < Z < \frac{15-10}{4}\right)$$
$$= P(0,5 < Z < 1,25)$$
$$= 0,3944 - 0,1915$$
$$= 0,2029$$

(b) $$P(X > a) = 0,08$$
$$1,405 = \frac{a-10}{4}$$
$$a = 15,62 \quad \therefore 16 \text{ minutes}$$

(c) $$P(X > 15 | X > 12) = \frac{P(X > 15)}{P(X > 12)}$$
$$= \frac{0,5 - 0,3944}{0,5 - 0,1915}$$
$$= 0,3423$$

(d) $$X \sim B(6;\, 0,2029)$$
$$P(X = 2) = \binom{6}{2}(0,2029)^2(0,7971)^4$$
$$= 0,2493$$

QUESTION 3

3.1 (a) A 96% CI for μ is:

$$\bar{x} = \frac{565}{220} = 2,57$$

$$2,57 \pm 2,05 \times \frac{1,45}{\sqrt{220}}$$

$$(2,37;\ 2,77)$$

(b) 3 is not in the confidence interval hence Naomi's claim is unjustified/ incorrect.

3.2 $H_0 : \mu_x - \mu_y = 140$

$H_1 : \mu_x - \mu_y > 140$

Reject H_0 if $Z > 1,645$

Test Statistic:

$$Z = \frac{(1752 - 1598) - 140}{\sqrt{\frac{50^2}{40} + \frac{20^2}{50}}}$$

$$= 1,67$$

Conclusion: Since $Z > 1,645$ reject the H_0 at the 5% level of significance and hence sufficient evidence to suggest the mean of X is greater than the mean of Y by at least 140.

QUESTION 4

(a) $p = 0,45$

(b) 0,25

(c) $(0,75)^2\,(0,25)$
$= 0,1406$

(d) $E[K] = 1(0,15) + 2(0,45) + 3(0,35) = 2,1$

$$\sigma^2 = 1(0,15) + 4(0,45) + 9(0,35) - (2,1)^2$$
$$\sigma = 0,8307$$

(e) $P(M + K > 4) = P(M = 2) \times P(K = 3) + P(M = 3) \times P(K = 2) + (M = 3) \times P(K = 3)$

$$= (0,45)(0,35) + (0,25)(0,45) + (0,25)(0,35)$$
$$= 0,3575$$

QUESTION 5

(a) The mode has the highest probability as it is the value that occurs the most often.

(b) $\int_0^4 kx^2\,dx = \frac{1}{2}$

$$\left[\frac{kx^3}{3}\right]_0^4 = \frac{1}{2}$$
$$\frac{64k}{3} = \frac{1}{2}$$
$$k = \frac{3}{128}$$

(c) $\frac{1}{2}(p-4)\left(\frac{3}{8}\right) = \frac{1}{2}$

$$p - 4 = \frac{8}{3}$$
$$p = \frac{20}{3}$$

QUESTION 6

(a) 6 or 8 _ _ _ _ _ _ _

$$\frac{7!}{2!2!2!}\times 2 = 1260 \quad \text{or} \quad \frac{1}{2}\times\frac{8!}{2!2!2!2!}$$

(b) 1st option: all 4 numbers are distinct (e.g. 6 842 – has to start with 6 or 8)

$$3!\times 2 = 12 \text{ or } \frac{1}{2}\times 4! = 12$$

$$\left(\text{by symmetry } \frac{1}{2} \text{ of all possible arrangements}\right) \text{ or } 2\times {}^3P_3 = 12$$

2nd option:1 double repeat and 2 distinct (e.g. 6 228 – has to start with 6 or 8)

6_ _ _ + 66_ _ + 8_ _ _ + 88_ _

$$= \left(3\times 1\times 2\times\frac{3!}{2!} + \binom{3}{2}\times 3!\right)\times 2 = 72$$

Or

$$\frac{1}{2}\times\frac{4!}{2!}\times 4\times\binom{3}{2} = 72$$

$$\left(\frac{1}{2} \text{ of all possible arrangements} \times \text{arrangement} \times \#\text{possible doubles} \times \text{selecting 2 from 3 other numbers}\right)$$

Option 3: 2 double repeats (e.g. 6622)

66 _ _ + 88_ _

$$3\times\frac{3!}{2!}\times 2 = 18$$

Or

$$\frac{1}{2}\times\frac{4!}{2!2!}\times\binom{4}{2} = 18$$

$$\left(\frac{1}{2} \text{ of all possible arrangements} \times \text{arrangement} \times \times \text{selecting 2 pairs from 4 numbers}\right)$$

Hence total # options:

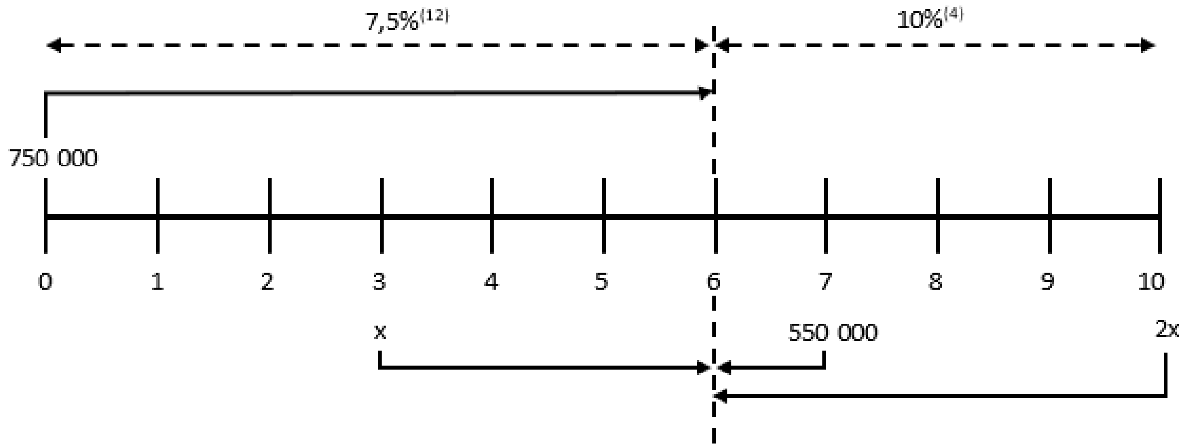
$$12 + 72 + 18 = 102$$

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

(a)



$$750\,000\left(1+\frac{0,075}{12}\right)^{72} = x\left(1+\frac{0,075}{12}\right)^{36} + 550\,000\left(1+\frac{0,1}{4}\right)^{-4} + 2x\left(1+\frac{0,1}{4}\right)^{-16}$$

$\therefore\ x = \text{R}260251{,}76$

(b) $550\,000\left(1+\frac{0,1}{4}\right)^{-4} + 2 \times 260251{,}76\left(1+\frac{0,1}{4}\right)^{-16}$

$= \text{R}\ 848\ 897{,}01$

[19]

QUESTION 2

$$\left(1+\frac{r}{2}\right)^2 = \left(1+\frac{0,08}{4}\right)^4$$

$$r = 0,0808$$

$$F = \frac{30\,000\left[\left(1+\frac{0,08}{4}\right)^{61}-1\right]}{\frac{0,08}{4}} + \frac{100\,000\left[\left(1+\frac{0,0808}{2}\right)^{30}-1\right]}{\frac{0,0808}{2}}$$

R9166092,92

QUESTION 3

(a) $P \times i = 15\,000$

$$i = \frac{15\,000}{1500\,000} = 0,01$$

$$1\,500\,000\,(1+0,01)^{11} = \frac{x\left[1-(1+0,01)^{-109}\right]}{0,01}$$

$$x = R25\,281,09$$

(b) $$O.B_{72} = 1\,500\,000\,(1+0,01)^{72} - \frac{25\,281,09\left[(1,01)^{61}-1\right]}{0,01}$$

$$= R960\,022,62$$

$$O.B_{84} = 1\,500\,000(1,01)^{84} - \frac{25\,281,09\left[(1,01)^{73}-1\right]}{0,01}$$

$$= R761\,150,02$$

$$\therefore \Delta OB = R198\,872,60$$

$\therefore$ Interest = 12 × 25 781,09 – 198 872,60

= 104 500,48

QUESTION 4

(a) (i) 0,08 (2)

(ii) $m = \dfrac{0{,}064 - 0{,}08}{400}$

$= -0{,}000\ 04$

$\therefore \dfrac{r}{k} = 0{,}000\ 04$

$\therefore k = \dfrac{0{,}08}{0{,}00004} = 2\ 000$

(iii) $\dfrac{\Delta P}{P} = -0{,}000\ 04\,P + 0{,}08$

$\therefore 0{,}032 = -0{,}000\ 04\,P + 0{,}08$

$\therefore P = 1\ 200$

(b) $P_{n+1} = P_n + 0{,}08\,P_n\left(1 - \dfrac{P_n}{2000}\right)$

(c) $\dfrac{2000 - 200}{2} = 900$ inhabitants

25 – 26 cycles / iterations

QUESTION 5

(a) $a = 0,6 \times 1 \times 1 \times 0,8$

$= 0,48$

(b) $20 \times 10 = 20 \times 500 \times b$

$\therefore b = 0,02$

(c) $f(0,02)(20)(500) = 15$

$\therefore f = 0,075$

(d) $$95 = \frac{c}{0,075 \times 0,02}$$

$\therefore c = 0,1425$

(e) $$20 = \frac{0,48}{0,02}\left(1 - \frac{0,1425}{0,015 \times 0,02 \times K}\right)$$

$K = 570$

QUESTION 6

(a) $9 = 5a - 3 + b$

$\therefore b = 12 - 5a$

$15 = 9a - 5 + b$

$b = 20 - 9a$

$\therefore a = 2$ and $b = 2$

(b) $T_5 = 2(15) - 9 + 2$

$= 23$

$T_6 = 2(23) - 15 + 2$

$= 33$

(c) Difference increases by 2 …

$\therefore T_n = T_n - 1 + 2(n - 1)$

$\therefore T_n = T_{n-1} + 2n - 2$

Total for Module 3: 100 marks

MODULE 4 GRAPH THEORY & MATRICES

QUESTION 1

1.2 Two columns are a repeat of each other, or volume is zero

1.3 (a) $A^T = \begin{bmatrix} 1 & -1 & -2 \\ 3 & 2 & 4 \\ 0 & -p & 0 \end{bmatrix}$

(b) $\det(A) = 0 - (-p)(4+6) + 0$ using column 3
$= 10p$

$\det(A^T) = 0 - (-p)(4+6) + 0$ using row 3
$= 10p$
$\therefore \det(A) = \det(A^T)$

(c) $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & -7 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 \\ 3 & 2 & 4 \\ 0 & -7 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 5 & 10 \\ 5 & 54 & 10 \\ 10 & 10 & 20 \end{pmatrix}$ for each row (column)

(d) Yes, as A is a reflection of A^T

QUESTION 2

$$x + 3y - 2z = -7$$
$$4x + y + z = 5$$
$$2x - 5y + 7z = 19$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 4 & 1 & 1 \\ 2 & -5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 19 \end{bmatrix}$$

$R_2 - 4R_1$; $R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 & : & -7 \\ 0 & -11 & 9 & : & 33 \\ 0 & -11 & 11 & : & 33 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 3 & -2 & : & -7 \\ 0 & -11 & 9 & : & 33 \\ 0 & 0 & 2 & : & 0 \end{bmatrix}$$

$z = 0$

$-11y = 33$
$y = -3$

$x + 3(-3) + 0 = -7$
$x = 2$

QUESTION 3

3.1 $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 4 & 4 & -1 \\ 1 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 4 & -1 \\ 2 & 2 & 4 & 4 \end{pmatrix}$

3.2 Area is 2 x the original.

3.3 $+\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$

3.4 $m = \frac{2}{3}$

tan $\theta = 2/3$ Rad = 0,588

$\theta = 33,69$

$\begin{bmatrix} \cos 33,69 & -\sin 33,69 \\ \sin 33,69 & \cos 33,69 \end{bmatrix}\begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ pre-multiplying

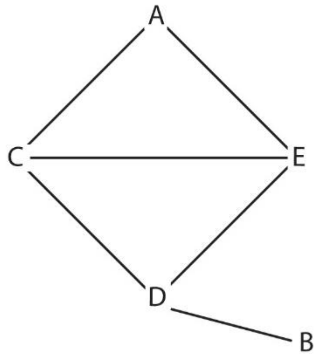
$\begin{pmatrix} -1,387 & 2,774 & 2,219 & -1,941 \\ 0,277 & 3,051 & 3,883 & 1,109 \end{pmatrix}$

QUESTION 4

4.1 (a)

	A	B	C	D	E
A	–		1		1
B		–		1	
C	1		–	1	1
D		1	1	–	1
E	1		1	1	–

(b)



Degree
Edges

(c) 4 points have odd degree

4.2 Forced to form a circuit

QUESTION 5

5.1 E – G 75 (compulsory)

G – D	30 (or ED)
G – F	35
F – A	40
E – B	45
G – C	65
	215

Cable length 290

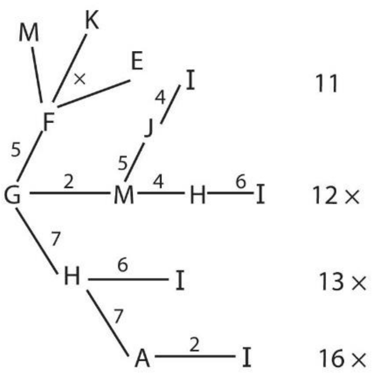
5.2 (a) Sum of edges = 111

Isolate odd degree edges A; C; E; G

Add in: CBA 10
EFG 10

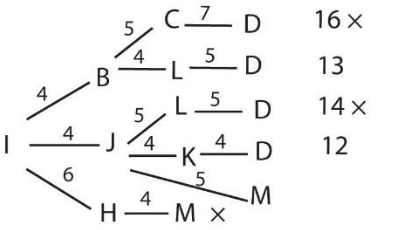
$\therefore$ shortest = 111 + 10 + 10 = 131

5.2 (b) Tourist ⟶ information



Information to toilet

G ⟶ M – J – I 11 units



I → J – K – D 12 units
$\therefore$ 23 units

QUESTION 6

6.1 $$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$

$$= \begin{pmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$\therefore$ B is nilpotent

6.2 (a) *P = 2*

(b) $$I - A = \begin{pmatrix} -1 & +2 & 4 \\ +1 & -2 & -4 \\ -1 & +2 & 4 \end{pmatrix}$$

$$(I - A)^2 = \begin{pmatrix} -1 & +2 & 4 \\ +1 & -2 & -4 \\ -1 & +2 & 4 \end{pmatrix} = I - A$$

OR

$$(I - A)^2 = I^2 - 2IA + A^2$$
$$= I - 2A + A \text{ (since } A^2 = A\text{)}$$
$$= I - A$$

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Total for Module 4: 100 marks