The Answer Series

Gr 12 Maths BOOKWORK

2023/2024

(Extracts from TAS Gr 12 Maths 2-in-1 and Toolkit)

All the proofs you need to know!

(also available in Afrikaans)

Paper 1: *(a maximum of 6 marks)* **page 1**

Arithmetic & Geometric Series

Paper 2: *(a maximum of 12 marks)* **page 2 to 7**

Geometry (7 theorems) and

Trigonometry (4 proofs & 5 formulae derivations)

Plus Calculator Instructions

THE ANSWER SERIES *Your Key to Exam Success*

PAPER 1: PROOFS

Arithmetic and Geometric Series

S_n: the sum of n terms

Arithmetic Series:

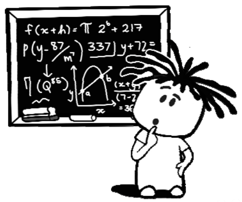
$$S_n = a + (a+d) + (a+2d) + \dots + T_n \quad \dots \text{ n terms}$$

$$\& \quad S_n = T_n + (T_n - d) + (T_n - 2d) + \dots + a \quad \dots \text{ n terms}$$

$$\therefore 2S_n = (a+T_n) + (a+T_n) + (a+T_n) + \dots + (a+T_n) \quad \dots \text{ n terms}$$

$$\therefore 2S_n = n(a+T_n)$$

$$\therefore \mathbf{S_n = \frac{n}{2}(a + T_n)}$$



But $T_n = a + (n-1)d$

$$\therefore S_n = \frac{n}{2}[a + a + (n-1)d]$$

$$\therefore \mathbf{S_n = \frac{n}{2}[2a + (n-1)d]}$$

Geometric Series:

$$S_n = a + ar + ar^2 + ar^3 + \dots\dots + ar^{n-2} + ar^{n-1} \quad \dots \ ❶$$

$$\times \mathbf{r}) \quad \therefore rS_n = \quad ar + ar^2 + ar^3 + \dots\dots \quad + ar^{n-1} + ar^n \quad \dots \ ❷$$

the 'middle bit' falls away.

❶ – ❷: $\therefore S_n - rS_n = a - ar^n$

$$\therefore S_n(1-r) = a(1-r^n)$$

$$\therefore \mathbf{S_n = \frac{a(1-r^n)}{1-r}} \quad \dots \textit{for } r < 1 \quad \textbf{or} \quad \times\frac{(-1)}{(-1)}: \ \mathbf{S_n = \frac{a(r^n-1)}{r-1}} \quad \dots \textit{for } r > 1$$

Notice that: $\dfrac{b-a}{d-c} = \dfrac{-(a-b)}{-(c-d)} = \dfrac{a-b}{c-d}$

Also remember:

- $S_\infty = S_n$ as $n \to \infty$ **if $-1 < r < 1$**

  $= \dfrac{a(1-\mathbf{0})}{1-r} \quad \dots r^n \to 0$ if $-1 < r < 1$

  $= \dfrac{a}{1-r}$

- $\displaystyle\sum_{k=1}^{n} T_k = S_n$ and $\displaystyle\sum_{k=1}^{\infty} T_k = S_\infty$ if $-1 < r < 1$

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PAPER 2: EXAMINABLE PROOFS 2023/2024

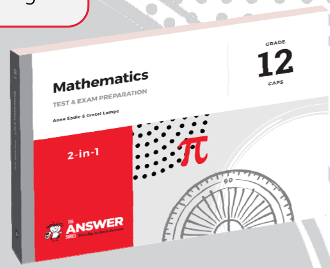
These are the examinable proofs required:

5 Gr 11 & 2 Gr 12

The Answer Series Mathematics publications have been designed to develop ...

- conceptual understanding
- reasoning techniques
- procedural fluency & adaptability
- a variety of strategies for problem-solving

Gr 12 Maths 2-in-1



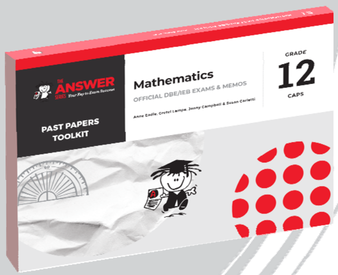
- Graded questions & detailed answers in Topics
- 14 CAPS exam papers with detailed solutions, **National** and **IEB**, **PLUS**
- Vital, supportive **documents**, **summaries** & **Topic Guides**
- **EXTENSION: Level 3 & 4 Questions** & Solutions

Ideal for use *throughout the year*, facilitating gradual, thorough concept development via its uniquely-designed question and answer route to mastery

Gr 12 PAST PAPERS TOOLKIT

This product is indeed a **'TOOLKIT'** containing **DBE** & **IEB** examination papers, detailed solutions and Topic Guides, plus vital, supportive **documents** and **summaries**.

Perfect for Exam Preparation for matrics!



EUCLIDEAN GEOMETRY

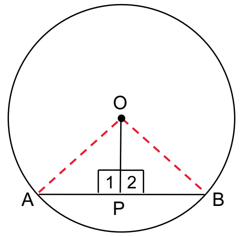
Circle Geometry Theorems

1 The line segment drawn from the centre of a circle, perpendicular to a chord, bisects the chord.

Given: $\odot$O with OP $\perp$ AB

To Prove: AP = PB

Construction: Join OA and OB



Proof: In Δ^s OPA & OPB

(1) OA = OB ... *radii*

(2) $\hat{P}_1 = \hat{P}_2$ (= 90º) ... *given*

(3) OP is common

$\therefore$ ΔOAP $\equiv$ ΔOBP ... *RHS*

$\therefore$ **AP = PB, i.e. OP bisects chord AB**

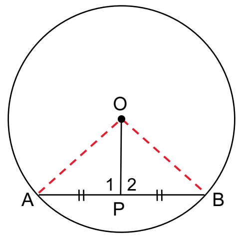
2 The line drawn from the centre of a circle that bisects a chord is perpendicular to the chord.

This proof has been added in the 2021 Exam Guidelines.

Given: $\odot$O with AP = PB

To Prove: OP $\perp$ AB

Construction: Join OA and OB



Proof: In Δ^s OPA & OPB

(1) OA = OB ... *radii*

(2) AP = PB ... *given*

(3) OP is common

$\therefore$ ΔOAP $\equiv$ ΔOBP ... *SSS*

$\hat{P}_1 = \hat{P}_2$

But, $\hat{P}_1 + \hat{P}_2 = 180º$ ... $\angle^s$ *on a straight line*

$\therefore$ $\hat{P}_1 = \hat{P}_2 = 90º$

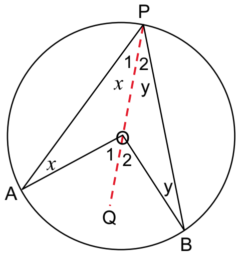
i.e. OP $\perp$ AB

3 The angle which an arc of a circle subtends at the centre is double the angle it subtends at any point on the circumference.

Given: ⊙O, arc AB subtending $A\hat{O}B$ at the centre and $A\hat{P}B$ at the circumference.

To Prove: $A\hat{O}B = 2A\hat{P}B$

Construction: Join PO and produce it to Q.



Proof: Let $\hat{P}_1 = x$

Then $\hat{A} = x$ . . . *∠s opposite equal radii*

$\therefore\ \hat{O}_1 = 2x$ . . . *exterior ∠ of ΔAOP*

Similarly, if $\hat{P}_2 = y$, then $\hat{O}_2 = 2y$

$\therefore$ **$A\hat{O}B$** $= 2x + 2y$

$= 2(x + y)$

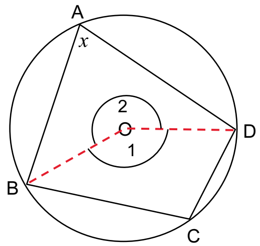
$=$ **$2\,A\hat{P}B$** ◂

4 The opposite angles of a cyclic quadrilateral are supplementary.

Given: ⊙O and cyclic quadrilateral ABCD

To Prove: $\hat{A} + \hat{C} = 180^\circ$ & $\hat{B} + \hat{D} = 180^\circ$

Construction: Join BO and DO.



Proof: Let $\hat{A} = x$

Then $\hat{O}_1 = 2x$ . . . *∠ at centre = 2 × ∠ at circumference*

$\therefore\ \hat{O}_2 = 360^\circ - 2x$ . . . *∠s round point O*

$\therefore\ \hat{C} = \frac{1}{2}(360^\circ - 2x) = 180^\circ - x$ . . . *∠ at centre = 2 × ∠ at circumference*

$\therefore$ **$\hat{A} + \hat{C} = x + 180^\circ - x = 180^\circ$**

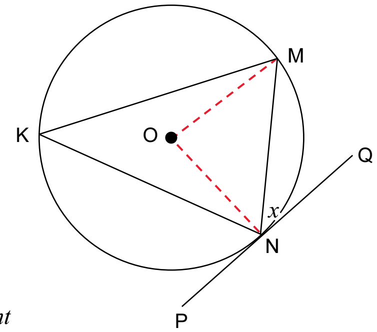
& $\therefore$ **$\hat{B} + \hat{D} = 180^\circ$** ◂ . . . *sum of ∠s in quad = 360°*

5 The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle subtended by the chord in the alternate segment.

Method 1

Draw radii and use '∠ at centre' theorem.

Given: ⊙O with tangent at N and chord NM subtending $\hat{K}$ at the circumference.



RTP: $M\hat{N}Q = \hat{K}$

Construction: radii OM and ON

Proof: Let $M\hat{N}Q = x$

$O\hat{N}Q = 90^\circ$ . . . *radius ⊥ tangent*

$\therefore\ O\hat{N}M = 90^\circ - x$

$\therefore\ O\hat{M}N = 90^\circ - x$ . . . *∠s opposite equal radii*

$\therefore\ M\hat{O}N = 2x$ . . . *sum of ∠s in Δ*

$\therefore\ \hat{K} = x$ . . . *∠ at centre = 2 × ∠ at circumference*

$\therefore$ **$M\hat{N}Q = \hat{K}$** ◂

These proofs are logical & easy to follow.

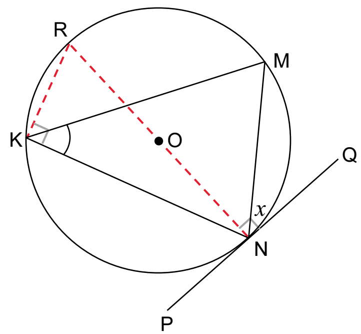
Method 2

We use **2** 'previous' **facts involving right ∠s**

❶ **tangent ⊥ diameter** . . . *so, draw a diameter!*

❷ **∠ in semi-⊙ = 90°** . . . *so, join RK!*

Given: ⊙O with tangent at N and chord NM subtending $\hat{K}$ at the circumference.



RTP: $M\hat{N}Q = M\hat{K}N$

Construction: diameter NR; join RK

Proof: $R\hat{N}Q =$ **90°** . . . *tangent ⊥ diameter*

& $R\hat{K}N =$ **90°** . . . *∠ in semi-⊙*

Then . . .

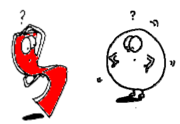
Let $M\hat{N}Q = \boldsymbol{x}$

$\therefore\ R\hat{N}M =$ **$90^\circ - x$**

$\therefore\ R\hat{K}M =$ **$90^\circ - x$** . . . *∠s in the same seg*

$\therefore\ M\hat{K}N = \boldsymbol{x}$

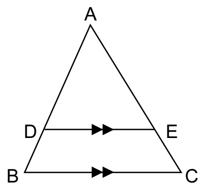
$\therefore$ **$M\hat{N}Q = M\hat{K}N$** ◂



▶ The Proportion Theorem

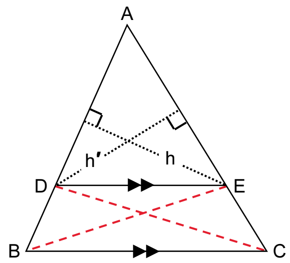
6 A line parallel to one side of a triangle divides the other two sides proportionally.

i.e. $DE \parallel BC \Rightarrow \dfrac{AD}{DB} = \dfrac{AE}{EC}$



Given: ΔABC with $DE \parallel BC$,
D & E on AB & AC respectively.

To prove: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$



Construction: Join DC & BE

Proof: $\dfrac{\text{Area of } \Delta ADE}{\textbf{Area of } \boldsymbol{\Delta DBE}} = \dfrac{\frac{1}{2}AD\,.\,h}{\frac{1}{2}DB\,.\,h} = \dfrac{AD}{DB}$

h is the height of Δ^s ADE and DBE

h′ is the height of Δ^s ADE and EDC

Similarly: $\dfrac{\text{Area of } \Delta ADE}{\textbf{Area of } \boldsymbol{\Delta EDC}} = \dfrac{AE}{EC}$ $\left(\dfrac{\frac{1}{2}AE\,.\,h'}{\frac{1}{2}EC\,.\,h'}\right)$

But: **ΔDBE = ΔEDC**, in area . . . *on the same base DE ; between || lines, DE & BC*

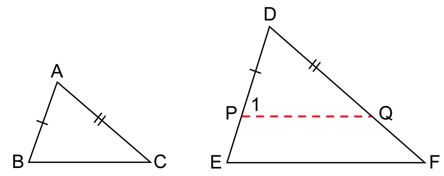
and: ΔADE is common

$\therefore \dfrac{\text{Area of } \Delta ADE}{\text{Area of } \Delta DBE} = \dfrac{\text{Area of } \Delta ADE}{\text{Area of } \Delta EDC}$

$\therefore \dfrac{AD}{DB} = \dfrac{AE}{EC}$ ◄

▶ The Similar Δ^s Theorem

7 If two triangles are equiangular, then their sides are proportional and, therefore, they are similar.



Given: ΔABC & ΔDEF with $\hat{A} = \hat{D}$ $\hat{B} = \hat{E}$ & $\hat{C} = \hat{F}$

To prove: $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$

Construction: Mark P & Q on DE & DF such that DP = AB & DQ = AC

Proof: In Δ^s DPQ & ABC

(1) DP = AB . . . *construction*
(2) DQ = AC . . . *construction*
(3) $\hat{D} = \hat{A}$ . . . *given*

$\therefore \Delta DPQ \equiv \Delta ABC$ . . . *S∠S*

stage 1: congruency

$\therefore \hat{P}_1 = \hat{B}$
$= \hat{E}$ . . . *given*

stage 2: corresponding $\angle^s$

The focal point

$\therefore$ PQ || EF . . . *corresponding $\angle^s$ equal*

$\therefore \dfrac{DP}{DE} = \dfrac{DQ}{DF}$ . . . *proportion theorem ; PQ || EF*

stage 3: parallel lines

But DP = AB and
DQ = AC . . . *construction*

$\therefore \dfrac{AB}{DE} = \dfrac{AC}{DF}$

stage 4: proportions

Similarly, by marking S and T on DE and EF such that SE = AB and ET = BC, it can be proved that: $\dfrac{AB}{DE} = \dfrac{BC}{EF}$

$\therefore \dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$ ◄

$\therefore$ ΔABC and ΔDEF are similar.

Similar Δ^s

Δ^s are similar if: **A:** they are equiangular, and
B: their sides are proportional

In this proof, we show that **A** ➡ **B**

$\therefore$ The Δ^s are similar . . . Both conditions, **A** and **B**, apply

The converse statement says: **B** ➡ **A**
$\therefore$ *The Δ^s are similar*

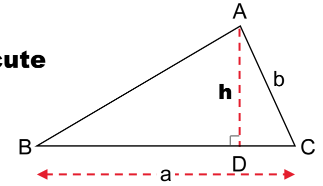
TRIGONOMETRY PROOFS

Area, Sine & Cosine Rules

① The Area rule

$$\boxed{\text{Area of } \triangle ABC = \frac{1}{2} ab \sin C}$$

$\hat{C}$ acute



CONSTRUCTION: Draw $AD \perp BC$

PROOF: Area of $\triangle ABC = \frac{1}{2} a\mathbf{h}$ . . . ❶

But, in $\triangle ACD$: $\frac{\mathbf{h}}{b} = \sin C$

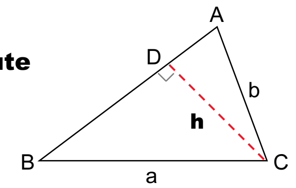
$\therefore\ \mathbf{h} = b \sin C$ . . . ❷

❷ in ❶: $\therefore$ **Area of $\triangle ABC = \frac{1}{2} ab \sin C$**

② The Sine rule

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b}} \boxed{= \frac{\sin C}{c}}$$

$\hat{B}$ acute



CONSTRUCTION: Draw $CD \perp AB$

PROOF: In $\triangle ADC$: $\frac{\mathbf{h}}{b} = \sin A$

$\therefore\ \mathbf{h} = b \sin A$ . . . ❶

In $\triangle BDC$: $\frac{\mathbf{h}}{a} = \sin B$

$\therefore\ \mathbf{h} = a \sin B$ . . . ❷

Equating ❶ & ❷: $\therefore\ b \sin A = a \sin B$

$\div ab)$ $\therefore\ \frac{\sin A}{a} = \frac{\sin B}{b}$

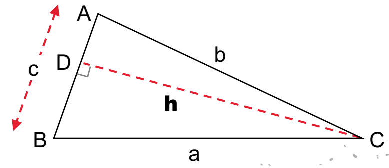
Similarly, by drawing a perpendicular from **B**, one can prove: $\frac{\sin A}{a} = \frac{\sin C}{c}$

$$\therefore\ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

③ The Cosine rule

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos A}$$

$\hat{A}$ acute



CONSTRUCTION: Draw $CD \perp BA$

PROOF: $a^2 = BD^2 + h^2$ . . . *Pythagoras*

$\therefore\ a^2 = (c - AD)^2 + h^2$

$= c^2 - 2c\,.\,AD + \underbrace{AD^2 + h^2}$

$= c^2 - 2c\,.\,AD + \quad b^2$ . . . *Pythagoras*

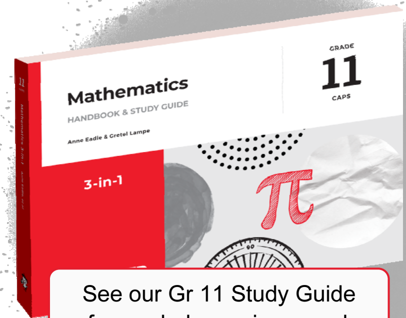
$= b^2 + c^2 - 2c\,.\,AD$ . . . ❶

In $\triangle ADC$: $\frac{AD}{b} = \cos A$

$\therefore\ AD = b \cos A$ . . . ❷

❷ in ❶:

$\therefore$ **$a^2 = b^2 + c^2 - 2bc \cos A$**



See our Gr 11 Study Guide for graded exercises and assessments.

For CONSTRUCTIONS in all 3 proofs,

Area Rule, Sine Rule & Cosine Rule

Always construct the height from a vertex NOT involved in the formula

- To prove: Area = $\frac{1}{2}$ab sin **C**, draw a height from **A** or **B**, not **C**.
- To prove: $\frac{\sin A}{a} = \frac{\sin B}{b}$, draw a height from **C**, not **A** or **B**.
- To prove: $a^2 = b^2 + c^2 - 2bc \cos A$, draw a height from **B** or **C**, not **A**.

Compound Angle Formulae

1. **sin(A + B) = sin A cos B + cos A sin B**
2. **sin(A – B) = sin A cos B – cos A sin B**

} *Sign stays the same sine & cosine of A and B mixed*

3. **cos(A + B) = cos A cos B – sin A sin B**
4. **cos(A – B) = cos A cos B + sin A sin B**

} *Sign changes cosine of A and B first, then sine of A & B*

- The **PROOF** of the formula: **cos(A – B) = cos A cos B + sin A sin B is examinable**. So, too, **derivations 1, 2 & 3** from this formula.

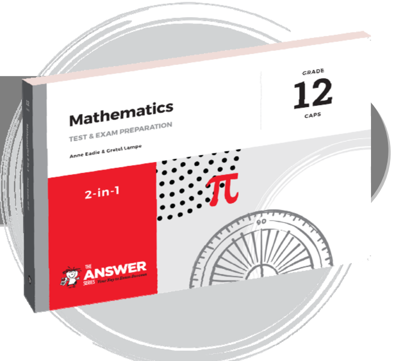
See our **TAS** Maths **CAPS** curriculum **Pp. 15** & **42** on our **TAS FET COMMUNITY PAGE**

Double Angle Formulae

5. **sin 2A = 2 sin A cos A** . . . Derived from formula **no. 1**.
6. **cos 2A = cos²A – sin²A** . . . Derived from formula **no. 3**.

or **cos 2A = 1 – 2 sin²A** or **cos 2A = 2 cos²A – 1**

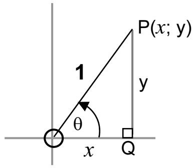
Refer to our Gr 12 Maths 2-in-1 (#9 P. 25)



Proof of the Formula:

cos(A – B) = cos A cos B + sin A sin B

NOTE: **First, an important concept!**



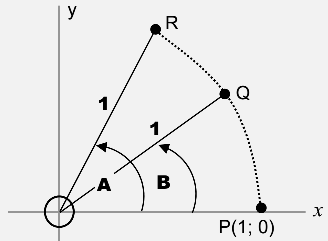
If OP = 1 unit!

then: $\frac{x}{1} = \cos\theta$ and $\frac{y}{1} = \sin\theta$

i.e. $x = \cos\theta$ and $y = \sin\theta$

i.e. **P is the point (cos θ; sin θ)**

See $\hat{A}$ and $\hat{B}$ placed in standard position alongside.



$\mathbf{R\hat{O}Q = \hat{A} - \hat{B}}$.

The coordinates of the points **R** and **Q**, both **1 unit** from the origin, are:

R(cos A ; sin A) & **Q(cos B ; sin B)** . . . See **NOTE** above

▶ **Determine two expressions for RQ²**

$RQ^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$ . . . **COSINE RULE**

$= 2 - 2\cos(A - B)$ . . . ❶

& $RQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$ . . . **DISTANCE FORMULA**

$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$

$= 2 - 2\cos A\cos B - 2\sin A\sin B$ ◀ . . . ❷ . . . $\sin^2\theta + \cos^2\theta = 1$

▶ **Equate the two expressions for RQ² above:**

❶ = ❷ $\therefore\ 2 - 2\cos(A - B) = 2 - 2\cos A\cos B - 2\sin A\sin B$

▶ **Subtract 2:** $\therefore\ -2\cos(A - B) = -2\cos A\cos B - 2\sin A\sin B$

▶ **Divide by –2** $\left(\text{or } \times \text{ by } -\frac{1}{2}\right)$: $\therefore$ **cos(A – B) = cos A cos B + sin A sin B** ◀

Derivation of Compound Angle Formulae

We accept and use the formula:

cos(A – B) = cos A cos B + sin A sin B

- **cos(A + B)** = cos [A – (–B)] — *The* **cos** *of the* **difference** *of the two angles*

  = cos A cos(–B) + sin A sin(–B)

  = cos A cos B + sin A (–sin B)

  = cos A cos B – sin A sin B

- **sin(A + B)** = cos[90° – (A + B)]

  = cos[90° – A – B]

  = cos[(90° – A) – B] — *The* **cos** *of the* **difference** *of the two angles*

  = cos(90° – A) cos B + sin(90° – A) sin B

  = sin A cos B + cos A sin B

- **sin(A – B)** = cos[90° – (A – B)]

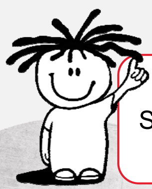
  = cos[90° – A + B]

  = cos[(90° – A) – (–B)] — *The* **cos** *of the* **difference** *of the two angles*

  = cos(90° – A) cos(–B) + sin(90° – A) sin(–B)

  = sin A cos B + cos A(–sin B)

  = sin A cos B – cos A sin B



Practice is essential!
See our graded exercises and copious exam questions in TAS Grade 12 study guides.

Derivation of Double Angle Formulae

- **sin 2A**

sin 2A = sin(A + A) . . . *see sin(A + B) above*

= sin A cos A + cos A sin A . . . *add like terms: xy + yx = 2xy*

= 2 sin A cos A

- **cos 2A**

cos 2A = cos(A + A) . . . *see cos(A + B) above*

= cos A cos A – sin A sin A

= $\cos^2 A - \sin^2 A$

But: $\sin^2 A + \cos^2 A = 1$

$\therefore \cos^2 A = 1 - \sin^2 A$ & $\sin^2 A = 1 - \cos^2 A$

$\therefore$ **cos 2A** = $(1 - \sin^2 A) - \sin^2 A$

= $1 - 2\sin^2 A$

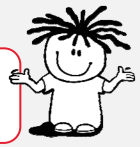
OR

cos 2A = $\cos^2 A - (1 - \cos^2 A)$

= $\cos^2 A - 1 + \cos^2 A$

= $2\cos^2 A - 1$

It is easier to memorise formulae which we understand.



CALCULATOR INSTRUCTIONS

A General Guide to Calculations

MEAN AND STANDARD DEVIATION

There are 3 phases for each calculator procedure:

STEP 1: How to get there

STEP 2: How to enter the data

STEP 3: How to find the **mean** and the **standard deviation**.
(and, in the next column, **A**, **B** and **r**)

Ungrouped Data

You'll see:

	X	FREQ
1	...	*If frequency is*
2	...	*on this*
3	...	*becomes 1 as*
4	...	*data is entered*

STEP 1:

- Press **MODE** ; Select **STAT** ;
  Select **1 – VAR**

STEP 2:

- **Enter each value**, followed by **=**.
  After the last value: **=** then press **AC** ← *

STEP 3:

- To find the **mean**: SHIFT **STAT** ; Select **Var** ;
  Select $\bar{x}$ **=**
- To find the **S.D.**: SHIFT **STAT** ; Select **Var** ;
  Select $x\sigma$**n** **=**

* → **Note**

It is essential to use **[AC]** *at the end of entering all the data.*

If this is not done, the mean and standard deviation will be added as entries in the data table.

Grouped Data/Frequency Tables

You'll see:

	X	FREQ
1	...	...
2	...	...
3	...	...

STEP 1:

- Press **MODE** ; Select **STAT** ;
  Select **1 – VAR**
- SHIFT **SETUP** ; Scroll down (use arrow)
  Select **STAT** ; Select **ON**

STEP 2:

- Enter the midpoint of each interval,
  followed by **=**, then . . .
  After the last value: **= (and not AC) then . . .**

  Use [arrow keys] to move to the top of the right-hand column.

  Type in the correct frequencies followed by **=** each time;
  after the last frequency: **= AC** ← *

STEP 3:

- To find the **mean**: SHIFT **STAT** ; Select **Var** ;
  Select $\bar{x}$ **=**
- To find the **S.D.**: SHIFT **STAT** ; Select **Var** ;
  Select $x\sigma$**n** **=**

REGRESSION & CORRELATION

The equation of the regression line $y = A + Bx$

You'll see:

	x	y	FREQ
1	...	...	...
2	...	...	...
3	...	...	...

STEP 1:
Press **MODE** ; Select **STAT** ;
Select **A + B**x

STEP 2:
Enter the x and y values, each followed by **=**

Use [arrow keys] to move to the top of the right-hand column.

After the last y value, press **=**, then press **AC**

STEP 3:
Press SHIFT **STAT** ; Select **Reg**

then:

Select **A**, press **=** or Select **B**, press **=**

To find **B** after **A**, press **AC** and then do the whole of **STEP 3**.

and

The correlation coefficient, r

Just like for **A** and **B** in the regression function:

FOLLOW STEPS 1, 2 & 3 above:

But, now: Select **r**, press **=**.

If you have found **A** and **B** first, press **AC** before going back to the beginning of **STEP 3**.

To clear completely: Press SHIFT **CLR** ; 1 = **AC**