INDEPENDENT EXAMINATIONS BOARD

INTERNATIONAL SECONDARY CERTIFICATE (IEB)

ASSESSMENT ADDENDUM

FURTHER STUDIES MATHEMATICS

Table of Contents

1. MEANS OF ASSESSMENT

A. Further Studies Standard: one 2-hour examination (Paper 1).

Further Studies Extended: two examinations, one 2-hour examination (Paper 1) and one 1 hour-examination (Paper 2).

Paper 1	2 hours	[200]
Paper 2	1 hour	[100]

B. Structure of Further Studies Mathematics: Standard & Extended

Module	Further Studies Standard Level	Further Studies Extended Level
CORE		
1A Calculus	✓	✓
1B Algebra	✓	✓
ELECTIVES		
2 Statistics & Probability **OR**		✓
3 Finance & Mathematical Modelling **OR**		✓
4 Matrices & Graph Theory		✓

C. Certification for Further Studies Mathematics:

Students will be awarded an ISC Further Studies (Standard) certificate if they register and offer Paper 1 **only** and attain 50% or above.

Students will be awarded an ISC Further Studies (Extended) certificate if they register and offer Paper 1 **and** Paper 2 and attain 50% or above in the **300-mark aggregate** of both papers.

- A Student who attains 50% or above in Paper 1 but because of Paper 2 does not achieve 50% or more in the **300-mark aggregate** will receive a Further Studies (Standard) subject certificate.
- A Student who attains 50% or above in Paper 2 but because of Paper 1 does not achieve 50% or more in the **300-mark aggregate** will not receive a Further Studies Mathematics pass but will receive a statement of results indicating the marks attained in both Paper 1 & Paper 2.

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2. EXAMINATION PAPER CONTENT REQUIREMENTS

PAPER 1

Module	Topic	Mark distribution (±5)
1A	Functions and limits	20
	Trigonometry	15
	Differentiation	35
	Integration	30
	Drawing functions	20
	Applications (max/min; rates of change; volume & area)	20
	Total	140
1B	Real and complex roots	15
	Exponents and logarithms	15
	Absolute value	20
	Induction	10
	Total	60

PAPER 2

Module	Topic	Percentage
2	Probability fundamentals	15–30
	Probability functions and applications	50–60
	Inferential statistics	15–30
OR	**Total**	100
3	Graph theory	40–60
	Matrices	40–60
OR	**Total**	100
4	Financial models	40–60
	Recursive models	40–60
	Total	100

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3. CALCULATORS

Candidates are expected to have a scientific calculator for use in the examinations. Programmable calculators and/or calculators with graphing capabilities, are not permitted.

4. WEIGHTING ACCORDING TO TAXONOMY OF COGNITIVE LEVEL FOR BOTH PAPER 1 AND PAPER 2

Examination Papers are designed to the following weighting:

Level		%
1	Knowledge	15 ± 5
2	Routine procedures	35 ± 5
3	Complex procedures	35 ± 5
4	Problem solving	15 ± 5
	Total	**100**

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5. INFORMATION/FORMULAE SHEET

INFORMATION BOOKLET

Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$|x| = \begin{cases} x & ; \quad x \geq 0 \\ -x & ; \quad x < 0 \end{cases}$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$z = a + bi \qquad z^* = a - bi$$

$$\ln A + \ln B = \ln (AB) \qquad \ln A - \ln B = \ln\left(\frac{A}{B}\right)$$

$$\ln A^n = n \ln A \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

Calculus

$$\text{Area} = \lim_{n \to \infty}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f\left(x_i\right)$$

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\int f'\left(g(x)\right).g'(x)\,dx = f(g(x)) + c$$

$$\int f(x).g'(x)\,dx = f(x).g(x) - \int g(x).f'(x)\,dx + c$$

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

$$V = \pi \int_a^b y^2 dx$$

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Function	**Derivative**
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sec x$	$\sec x.\tan x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x.\cot x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$f(g(x))$	$f'(g(x)).g'(x)$
$f(x).g(x)$	$g(x).f'(x)+f(x).g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$

$A=\frac{1}{2}r^2\theta$ $\qquad$ $s=r\theta$

In $\triangle ABC$: $\qquad$ $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$$a^2=b^2+c^2-2bc.\cos A$$

$$Area=\frac{1}{2}ab.\sin C$$

$\sin^2 A+\cos^2 A=1$ $\qquad$ $1+\tan^2 A=\sec^2 A$ $\qquad$ $1+\cot^2 A=\mathrm{cosec}^2 A$

$\sin(A\pm B)=\sin A.\cos B\pm\cos A\sin B$ $\qquad$ $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\sin 2A=2\sin A\cos A$ $\qquad$ $\cos 2A=\begin{cases}\cos^2 A-\sin^2 A\\ 2\cos^2 A-1\\ 1-2\sin^2 A\end{cases}$

$$\sin A.\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\sin A.\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A.\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

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Matrix Transformations

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Finance & Modelling

$$F = P(1+in) \qquad F = P(1-in) \qquad F = P(1+i)^n \qquad F = P(1-i)^n$$

$$F = x\left[\frac{(1+i)^n - 1}{i}\right] \qquad P = x\left[\frac{1-(1+i)^{-n}}{i}\right] \qquad r_{eff} = \left(1+\frac{r}{k}\right)^k - 1$$

$$P_{n+1} = P_n + rP_n\left(1-\frac{P_n}{K}\right)$$

$$R_{n+1} = R_n + aR_n\left(1-\frac{R_n}{K}\right) - bR_nF_n \qquad F_{n+1} = F_n + f.bR_nF_n - cF_n$$

Statistics

$$P(A) = \frac{n(A)}{n(S)} \qquad P(B\,|\,A) = \frac{P(B\cap A)}{P(A)} \qquad P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$^nP_r = \frac{n!}{(n-r)!} \qquad ^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} \qquad P(X=x) = \binom{n}{x}p^x(1-p)^{n-x}$$

$$P(R=r) = \frac{\binom{p}{r}\binom{N-p}{n-r}}{\binom{N}{n}} \qquad E[X] = n\cdot p \qquad Var[X] = n\cdot p(1-p)$$

$$z = \frac{X-\mu}{\sigma} \qquad z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \qquad z = \frac{(\bar{x}-\bar{y})-(\mu_x-\mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}}$$

$$\bar{x} \pm z\frac{\sigma}{\sqrt{n}} \qquad p \pm z\sqrt{\frac{p(1-p)}{n}} \qquad E[X] = \sum x\cdot P(X=x)$$

$$Var[X] = E\left[X^2\right] - \left(E[X]\right)^2$$

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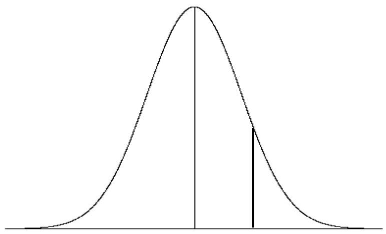
NORMAL DISTRIBUTION TABLE

Areas under the Normal Curve



$$H(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}x^2}\, dx$$

$H(-z) = H(z)$, $H(\infty) = \frac{1}{2}$

Entries in the table are values of $H(z)$ for $z \geq 0$.

z	,00	,01	,02	,03	,04	,05	,06	,07	,08	,09
0,0	,0000	,0040	,0080	,0120	,0160	,0199	,0239	,0279	,0319	,0359
0,1	,0398	,0438	,0478	,0517	,0557	,0596	,0636	,0675	,0714	,0753
0,2	,0793	,0832	,0871	,0910	,0948	,0987	,1026	,1064	,1103	,1141
0,3	,1179	,1217	,1255	,1293	,1331	,1368	,1406	,1443	,1480	,1517
0,4	,1554	,1591	,1628	,1664	,1700	,1736	,1772	,1808	,1844	,1879
0,5	,1915	,1950	,1985	,2019	,2054	,2088	,2123	,2157	,2190	,2224
0,6	,2257	,2291	,2324	,2357	,2389	,2422	,2454	,2486	,2517	,2549
0,7	,2580	,2611	,2642	,2673	,2704	,2734	,2764	,2794	,2823	,2852
0,8	,2881	,2910	,2939	,2967	,2995	,3023	,3051	,3078	,3106	,3133
0,9	,3159	,3186	,3212	,3238	,3264	,3289	,3315	,3340	,3365	,3389
1,0	,3413	,3438	,3461	,3485	,3508	,3531	,3554	,3577	,3599	,3621
1,1	,3643	,3665	,3686	,3708	,3729	,3749	,3770	,3790	,3810	,3830
1,2	,3849	,3869	,3888	,3907	,3925	,3944	,3962	,3980	,3997	,4015
1,3	,4032	,4049	,4066	,4082	,4099	,4115	,4131	,4147	,4162	,4177
1,4	,4192	,4207	,4222	,4236	,4251	,4265	,4279	,4292	,4306	,4319
1,5	,4332	,4345	,4357	,4370	,4382	,4394	,4406	,4418	,4429	,4441
1,6	,4452	,4463	,4474	,4484	,4495	,4505	,4515	,4525	,4535	,4545
1,7	,4554	,4564	,4573	,4582	,4591	,4599	,4608	,4616	,4625	,4633
1,8	,4641	,4649	,4656	,4664	,4671	,4678	,4686	,4693	,4699	,4706
1,9	,4713	,4719	,4726	,4732	,4738	,4744	,4750	,4756	,4761	,4767
2,0	,4772	,4778	,4783	,4788	,4793	,4798	,4803	,4808	,4812	,4817
2,1	,4821	,4826	,4830	,4834	,4838	,4842	,4846	,4850	,4854	,4857
2,2	,4861	,4864	,4868	,4871	,4875	,4878	,4881	,4884	,4887	,4890
2,3	,48928	,48956	,48983	,49010	,49036	,49061	,49086	,49111	,49134	,49158
2,4	,49180	,49202	,49224	,49245	,49266	,49286	,49305	,49324	,49343	,49361
2,5	,49379	,49396	,49413	,49430	,49446	,49461	,49477	,49492	,49506	,49520
2,6	,49534	,49547	,49560	,49573	,49585	,49598	,49609	,49621	,49632	,49643
2,7	,49653	,49664	,49674	,49683	,49693	,49702	,49711	,49720	,49728	,49736
2,8	,49744	,49752	,49760	,49767	,49774	,49781	,49788	,49795	,49801	,49807
2,9	,49813	,49819	,49825	,49831	,49836	,49841	,49846	,49851	,49856	,49861
3,0	,49865	,49869	,49874	,49878	,49882	,49886	,49889	,49893	,49896	,49900
3,1	,49903	,49906	,49910	,49913	,49916	,49918	,49921	,49924	,49926	,49929
3,2	,49931	,49934	,49936	,49938	,49940	,49942	,49944	,49946	,49948	,49950
3,3	,49952	,49953	,49955	,49957	,49958	,49960	,49961	,49962	,49964	,49965
3,4	,49966	,49968	,49969	,49970	,49971	,49972	,49973	,49974	,49975	,49976
3,5	,49977									
3,6	,49984									
3,7	,49989									
3,8	,49993									
3,9	,49995									
4,0	,49997									

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6. ELABORATION OF CONTENT

There are three levels and represent the progression of teaching and learning in each content topic.

Topic: Calculus (Module 1A)

The student is able to establish, define, manipulate, determine, and represent the derivative and integral, both as an anti-derivative and as the area under a curve, of various algebraic and trigonometric functions, and solve related problems.

	Sub-Topics		
	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
1		(a) Sketch the graphs of mathematical functions including split domain and composite functions (comprised of linear, quadratic, hyperbolic, absolute value, and exponential functions). (b) Manipulate, analyse split domain, and composite functions using the definition of a function and the graph of the function.	
2		(a) Define and use trigonometric and reciprocal trigonometric functions to: • manipulate trigonometric statements. • solve trigonometric problems in realistic and mathematical contexts. • find the general solution of trigonometric equations. (b) convert between angles measured in degrees and radians. (c) use trigonometric functions defined in terms of a real variable (angle in radians) and the *x*, *y,* and r-definition to: • calculate the lengths of arcs of circles, • calculate the area of sectors and segments of circles.	

	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
3		(a) Compare the graphical, numerical, and symbolic representations of the limit of a function. (b) Determine the limit of an algebraic function at a point, including from the right and left, and to infinity algebraically. (Note: The limit at a point is defined as the limit from the left and from the right.) (c) Illustrate the continuity of a function graphically and apply the definition of continuity at a point to simple algebraic functions, including split domain functions.	(a) Use first principles and graphs to determine the continuity and differentiability at a given point of algebraic functions, including split domain functions. (b) Without proof, apply the theorem and its converse, "A function that is differentiable at a point is continuous at that point", and deal with examples to indicate that the converse is not valid. (c) Demonstrate the derivative of a function at a point as the rate of change, by graphical, numerical, and symbolic representations.
4		(a) Illustrate the differentiability of a function graphically and determine the derivative as the gradient of a function at a point using limits. (b) Establish the derivatives of functions of the form $ax^2 + bx + c$ & $mx + c$, from first principles.	(a) Establish the derivatives of functions of the form $\sqrt{x}$, $\sqrt{ax+b}$, $\frac{1}{ax+b}$, $\frac{1}{\sqrt{ax+b}}$ from first principles. (b) Use the following rules of differentiation: $\frac{d}{dx}\left[f(x).(gx)\right] = g(x).\frac{d}{dx}\left[f(x)\right] + f(x).\frac{d}{dx}\left[g(x)\right]$ $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).\frac{d}{dx}\left[f(x)\right] - f(x).\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$ $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)).g'(x)$

	Level 1	Level 2	Level 3
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
			applied to: • polynomial, rational, radical, natural logarithm, ef(x) and trigonometric functions, (Students may assume without proof that $\frac{d}{dx}\sin x = \cos x$), • higher order derivatives, but the n-th derivative is not examinable, and • functions in two variables using implicit differentiation. (c) Newton's method. (d) Calculate maximum and minimum values of a function using calculus methods. Determine both absolute and relative maximum and minimum values of a function over a given interval. (e) Use calculus methods to sketch the curves of polynomial and rational functions determining: • intervals over which a function is increasing or decreasing. • *y*-intercepts and *x*-intercepts, using Newton's Method if necessary. • the coordinates and nature of stationary points. • any vertical, horizontal, or oblique asymptotes. (f) Use methods learned above to solve practical problems involving optimisation and rates of change in real, realistic, and abstract mathematical contexts. Verify the results of the calculus modelling by referring to the practical context.

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	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
5	(a) Approximate the area between familiar curves, such as straight lines, parabolas, hyperbolae and exponential graphs, and the *x*-axis using the Rectangle Method on an interval of the *x*-axis. (b) Estimate the margin of error of the approximate area determined by the upper and lower sums method. (c) Experiment with the accuracy of the approximation by varying the width and number of rectangles. (Not examinable in grade 12)	(a) Investigate and develop a formula for the upper and lower sums method of approximating area under the curve of $y = x^n$, for $n \in$ N and $x \geq 0$ on the interval [a; b] $\int_a^b x^n \, dx = \left[\frac{1}{n+1} x^{n+1}\right|_a^b$ (b) Use available technology to manipulate the width of sub-intervals and the accuracy of the approximate area under a polynomial function. (c) Investigate and intuitively develop the Riemann (definite) integral as the approximating rectangles are made narrower and the number of rectangles, *n*, increases. (Generating and simplifying the formula $Area = \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f\left(x_i\right)$ is not examinable. Students may be required to interpret the formula.)	(a) Appreciate the Fundamental Theorem of Calculus and its significance in recognising anti-differentiation as the reverse of differentiation. (b) Manipulate and then integrate algebraic, natural logarithm, $e^{f(x)}$ and trigonometric functions of the form: • $\int ax^n dx$, *a* is a constant and $n \in$ Q, • $\int p(x)\,dx$ and $\int \frac{p(x)}{q(x)}\,dx$, *p*(*x*) and *q*(*x*) are polynomials or radicals, • $\int \frac{f(x)}{g(x)}\,dx$, leading to partial fractions • $\int f(g(x)).g'(x)\,dx$ • where the anti-derivative of a trigonometric function can be directly determined from the derivative (e.g. $\int \cos x dx$, $\int \sec^2 x dx$, $\int \sec x \tan x dx$) • $\int \sin^n x dx$ and $\int \cos^n x dx$ where n $\in$ N and n $\leq$ 3 • $\int \sin mx \, dx$; $\int \sin mx \cos nx dx$ and similar functions

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	Level 1	Level 2	Level 3
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
			using only the following methods: • direct anti-derivatives • simplification of trigonometric functions using appropriate squares, compound angle and given product-sum formulae • integration by u-substitution • integration with a given substitution • integration by parts (c) Find the definite integral of any function using a calculator. (d) Apply the definite integral and techniques of integration to solve area and volume problems by: • Calculating the area under or between curves using the manipulation of intervals. • Calculating the volume generated by rotating a function about the x-axis in mathematical and real-world contexts.

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Topic: Algebra (Module 1B)

The student is able to represent, investigate, analyse, manipulate, and prove conjectures about numerical and algebraic relationships and functions, and solve related problems.

	Sub-Topics		
	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
6	(a) Characterise and discuss the nature and relevance of the roots of $x^2+1=0$. (b) Classify numbers using sub-fields of the Complex numbers. (c) Determine the roots of equations of the form $ax^2+bx+c=0$ and classify the roots as real or imaginary. Determine the real and complex roots of quadratic and cubic equations using: • factorisation, • the quadratic formula, and • the factor theorem to find the first real root of cubic equations. (d) Perform the four basic operations (+, –, /, x) on Complex numbers and their conjugates without the use of a calculator. (e) Illustrate a Complex number on an Argand diagram.	Solve: (a) equations containing multi-term algebraic fractions using algebraic methods. (b) polynomial and rational inequalities. (c) absolute value equations.	Solve exponential equations using the laws of exponents, algebraic manipulation, and logs.

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	Level 1	Level 2	Level 3
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
7	Simplify compound fractions.	Decompose algebraic fractions into partial fractions when the denominator is of the form: • $(a_1x+b_1)(a_2x+b_2)\ \dots\ (a_nx+b_n)$, using the 'cover up' method. • $(ax+b)^2(cx+d)^2$, by comparing coefficients.	(a) Simplify and manipulate algebraic expressions using the laws of exponents and logarithms. (b) Demonstrate an understanding of *e* and its role in exponents and logarithms by using *e* freely in problem solving.
8	(a) Demonstrate an understanding of the absolute value of an algebraic expression as a distance from the origin. $\lvert x\rvert = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ (b) Solve linear absolute value equations. (c) Draw graphs of the form $y = a\lvert x\rvert + q$.	(a) Draw graphs of $y = a\lvert x-p\rvert + q$, $y = \lvert f(x)\rvert$ and $y = f(\lvert x\rvert)$ by inference where $f(x)$ is a simple function (e.g. see 11.1.1). (b) Find the equation of the absolute value function, in the form $y = a\lvert x-p\rvert + q$, given the graph and necessary points on the graph. (c) Interpret the graphs of absolute value functions to determine the: • domain and range; • intercepts with the axes; • turning points, minima and maxima; • shape and symmetry; • transformations. (d) Solve absolute value inequalities using graphical representations of the associated functions.	(a) Draw exponential graphs, including $y = e^x$. (b) Draw logarithmic graphs, including y = ln x. (c) Find the equation for the reflections of the exponential or logarithmic functions in the lines x = 0, y = 0 and y = x, the inverse of the function. (d) transformations of ln(x) / e^x graphs.
9			Use mathematical induction to prove: (a) statements of summation of series. (b) statements about factors and factoring with Natural numbers.

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Topic: Statistics and Probability (Module 2)

The student is able to organise, summarise, analyse, and interpret data to identify, formulate and test statistical and probability models, and solve related problems.

	Sub-Topics		
	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
1	(a) Use Venn diagrams as an aid to solve probability problems of random events. (b) Use Tree diagrams as an aid to solve probability problems of random events. (c) Use contingency tables to solve probability problems. (d) Recognise and then determine the probability of conditional events using diagrams and the formula: $P(A\|B)=\frac{P(A\cap B)}{P(B)}$ (e) Identify and determine the probability of mutually exclusive and independent events. (f) Use the Laws of Probability to evaluate simple random events.	(a) Identify, apply, and calculate the probability of the following distributions of discrete random events: • Hypergeometric distribution model • Binomial distribution model The mean and variance of the binomial distribution should be known and applied. (b) Formulate and/or apply a probability mass function (including simple continuous models) and find the expected value and variance. (c) Identify and apply the Normal distribution model to the probability of continuous random events, using statistical tables and calculations, as necessary.	(a) Understand how the Central Limit Theorem is applied when a large ($n > 30$) sample is taken from any distribution. (b) Apply the Normal distribution model to a sample to estimate a confidence interval for the population mean or proportion, using statistical tables to deal with various confidence levels. (c) Apply the normal approximation to the binomial distribution, utilising continuity corrections, as appropriate. (d) Formulate a probability mass or density function for a: • Hypergeometric distribution • Binomial distribution • Simple continuous probability models
2	Count arrangements using permutations (including those where repetition occurs).	Count arrangements and choices using permutations (including those where repetition occurs) and combinations.	–

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	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
3	(a) Calculate the standard deviation. (b) Understand the mean and standard deviation of a population as applied to the normal distribution. The effect of these parameters on the shape of the curve should be understood. (c) The percentage of values that lie within 1, 2 and 3 standard deviations of the mean should be known.		Perform a one-tail and/or two-tail hypothesis test on a mean or difference of means. This includes being able to: • Distinguish between one-tail and two-tail events. • Establish a null hypothesis based on the prevalent condition.

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Topic: Finance and Mathematics Modelling (Module 3)

The student is able to investigate, represent and model growth and decay problems using formulae, difference equations and series.

	Sub-Topics		
	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
1	(a) Generalise number patterns using first order linear difference equations of the form $u_n = k.u_{n-1} + c$. (b) Use appropriate technology to solve higher terms in first order linear difference equations. (c) Convert first order linear difference equations into a general solution in explicit form. (d) With the aid of appropriate technology, use first order linear difference equations to solve future and present value annuities.	(a) Generalise or produce number patterns using second order homogenous linear difference equations $(u_n = p.u_{n-1} + q.u_{n-2} + c)$. (b) Use appropriate technology to solve higher terms in second order homogenous linear difference equations. (c) Model simple population growth and decay problems using a discrete Malthusian population model of the form $P_{n+1} = (1+r).P_n + c$.	(a) Model simple population growth and decay problems using • a discrete Logistic population model of the form $P_{n+1} = P_n + a(1-b.P_n).P_n$ • a discrete two species Lotka-Volterra predator-prey population model written in difference equation form $R_{n+1} = R_n + a.R_n(1- R_n / k) - b.R_n.F_n$ $F_{n+1} = F_n + e.b.R_n.F_n - c.F_n$ and interpret characteristics of the parameters and attributes of the models (e.g., carrying capacity, equilibrium populations) (b) Evaluate a realistic population scenario and apply the most suitable model for a given scenario.

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	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
2	(a) Use simple and compound growth formulae to solve problems in various contexts including but not limited to: • simple interest and straight-line depreciation, • compound interest and reducing balance depreciation, • compound growth and decay problems (b) Investigate and derive the future value annuity formula using first order linear difference equations in explicit form.	Formulate timelines and apply future and present value annuity formulae to: (a) Calculate the present value or future value of an annuity, or the termly payment. (b) Calculate the balance outstanding on a loan at a specified point in the amortisation period. (c) Establish a sinking fund in a given context. (d) Calculate the value of a deferred annuity. (e) Convert effective and nominal interest rates to solve problems with different accumulation periods.	Formulate timelines and apply future and present value annuity formulae to: (a) Determine the number of repayment periods using logarithms. (b) Calculate the number of payments and the final payment when a loan is repaid by fixed instalments. (c) Solve annuity problems involving changing circumstances such as changes to time periods, repayments (including missed payments), withdrawals and interest rates.

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Topic: Matrices and Graph Theory (Module 4)

The student is able to identify, represent and manipulate discrete variables using graphs and matrices, applying algorithms in modelling finite systems.

	Sub-Topics		
	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
1	(a) Arrange numbers in a suitable rectangular array or matrix to facilitate problem solving. (b) Knowing when a matrix operation is possible, perform the following operations on a matrix or matrices: • addition, • multiplication of a matrices, and • multiplication by a scalar. (c) Solve systems of two variable linear equations using the method of diagonalisation. (d) Determine the inverse of 2 × 2 matrices by a sequence of row transformation $[A : I_{nxn}] = [I_{nxn} : A^{-1}]$. (e) Solve systems of linear equations using the inverse matrix.	Use 2 × 2 matrices to transform points and figures in the Cartesian Plane by: (a) A translation, given in the form $\begin{pmatrix} a \\ b \end{pmatrix}$. (b) Rotation through any given angle about the origin. (c) Reflection in any given line through the origin. (d) Enlargement, using construction, with positive or negative scale factors and the centre of enlargement at the origin. (e) Shear and stretch with the *x* or *y* axis as the invariant line using negative or positive shear/stretch factors.	Use matrices to: (a) Solve systems of three variable linear equations using the method of diagonalisation (i.e., Gaussian Reduction). (b) Determine the inverse matrix by a sequence of row transformations using $[A : I_{nxn}] = [I_{nxn} : A^{-1}]$. (c) Calculate the determinant of the matrix. (d) Solve systems of linear equations using the inverse matrix up to a 4 by 4 matrix.

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	Level 1	**Level 2**	**Level 3**
	We know this when the student is able to:	We know this when the student is able to:	We know this when the student is able to:
2	(a) Define simple, regular, and connected graphs, their vertices, edges, and the degree of the graph. (b) Identify complete, complementary, and isomorphic graphs. (c) Define walks, paths, trails, cycles and circuits. (d) Evaluate and determine Eulerian paths within a graph. (e) Evaluate and classify graphs, intuitively and algorithmically, as Eulerian Circuits or Hamiltonian Circuits. (f) Use Euler's, Fleury's, and Dirac's algorithms to test the nature of the paths and circuits in a graph.	(a) Determine the number of different graphs that can be drawn on $n \leq 6$ vertices. (b) Represent simple network problems using a weighted graph. (c) Solve simple optimisation problems using weighted graphs. (d) Determine the shortest path of a network or weighted graph using Dijkstra's shortest path algorithm. (e) Optimise route inspection (Chinese postman) problems using Eulerian circuits, paths and the shortest path algorithm.	(a) Solve minimum connector problems using graphs, matrices and the Kruskal and Prim algorithms. (b) Solve by finding an upper bound for simple travelling salesman problems using graphs and matrices and the nearest-neighbour algorithm. (c) Solve simple travelling salesman problems using simple algorithms researched in the literature. (d) Use matrices to represent graphs and to solve optimisation problems.

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MATRICES: TERMINOLOGY AND NOTATION

M^{-1}:	inverse of a matrix
M^T:	transpose of a matrix
|M| or det M:	determinant of a matrix
Leading/main diagonal:	running from top left to bottom right
Trace:	the sum of the leading diagonal of a matrix
Order:	number of vertices in a graph
Size:	number of edges in a graph
Degree:	number of edges leading out of a vertex
Neighbourhood:	vertices directly connected a to given vertex
Singular matrix:	a matrix that has no inverse
Non-singular matrix:	a matrix that has an inverse
Row Echelon Form:	lower half of matrix below leading diagonal consists only of zero elements
Reduced Row Echelon Form:	matrix reduced to only zero elements, except main diagonal which is only ones

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GRAPH THEORY: TERMINOLOGY

A **graph** is a set of **vertices** and **edges;** every edge starts and finishes at a vertex.

A **connected** graph has all vertices directly or indirectly connected to each other.

A **complete** graph is a graph in which each pair of vertices is connected by an edge.

The **complement** of a graph consists of the same set of vertices but the edges in the complement are the edges not present in the original graph.

Simple graphs have no loops or multiple edges.

An **undirected** graph means that the distance A → B = B → A

The **order** of a graph is the number of vertices in the graph.

The **degree** of a vertex is the number of edges leading to/from that vertex.

The **size** of a graph is the total number of edges in the graph.

A **regular graph** has all vertices of the same degree.

Adjacent vertices are directly connected to each other (share a common edge).

The **neighbourhood** of a vertex are those vertices to which it is directly connected.

In a simple graph a **walk** is a sequence of vertices and edges in a graph. such that the finishing vertex of an edge becomes the starting vertex of the next edge. The edges and vertices need not distinct.

A **path** is a route in a graph so that no vertex is visited more than once; not all the vertices need be visited, and the starting point need not be the endpoint.

A **trail** is a route in a graph so that no edge is travelled more than once.

A trail that starts and ends at the same vertex is a **closed trail, circuit.** A **tree** is a connected graph, with **no circuits.**

The **weight** of an edge is the distance, time or cost factor ascribed to it.

Graphs can also be represented in **adjacency matrices** or as **geometric figures**.

Isomorphic graphs have the same size, order, and neighbourhoods (circuitry).

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GRAPH THEORY: AFRIKAANS TERMINOLOGY

Chinese Postman Problem	Chinese Posmanprobleem
Circuit/Ring	Kring
Coincident Planes	Samevallende Vlakke
Complete Graph	Volledige Grafiek
Complementary Graph	Komplimentêre Grafiek
Connected Graphs	Samehangende Grafieke
Edges of a Graph	Skakels van 'n Grafiek
Eulerian Circuit/Path	Eulerkring of -pad
Gaussian Reduction	Gaussherleiding
Girth of a Circuit	Gordellengte van 'n Kring
Hamiltonian Circuit	Hamiltonkring
Loop at Vertex A	Lus by Nodus A
Multigraph	Veelvoudige Grafiek
Nearest Neighbour Algorithm	Naaste Buurpunt Algoritme
Path between Vertices A and B	Pad tussen Nodusse A en B
Simple Graph	Enkelvoudige Grafiek
Spanning Tree	Spanboom
Travelling Salesman Problem	Reisende Verkoopsmanprobleem
Undirected Graph/Digraph	Ongerigte Grafiek
Upper and Lower Bounds	Bo- en Ondergrense
Weighted Graphs	Geweegde Grafieke
Adjoint Matrix	Adjunk Matriks
Adjacency Matrix	Nodus Matriks
Augmented Matrix	Aangevulde Matriks
Enlargement	Vergroting
Invariant Line for Transformations	Invariantelyn vir Transformasies
Inverse Matrix	Inverse Matriks
Leading Diagonal of Matrix	Hoofdiagonal van Matriks
Matrix of Cofactors	Kofaktormatriks
Matrix of Minors	Minormatriks
Reflection	Refleksie
Rotation	Rotasie
Row Reduction	Ry Reduksie
Shear	Dwarsdruk
Stretch	Rek
Translation	Translasie
Transpose of Matrix	Transponeer van Matriks
Vertex/Node of a Graph	Nodus van 'n Grafiek

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