INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2022

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

Time: 1 hour | 100 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 12 pages and an Information Booklet of 4 pages (i–iv). Please check that your question paper is complete.

2. This question paper consists of THREE modules.

   Choose **ONE** of the **THREE** modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

4. All necessary calculations must be clearly shown and writing must be legible.

5. Diagrams have not been drawn to scale.

6. **Rounding of final answers.**

   MODULE 2: Four decimal places, unless otherwise stated.

   MODULE 3: Two decimal places, unless otherwise stated.

   MODULE 4: Two decimal places, unless otherwise stated.

MODULE 2 STATISTICS

QUESTION 1

1.1 In a particular area, 75% of the people have been vaccinated against Covid-19. It is further known that 40% of the vaccinated had a previous infection of Covid-19 and 60% of the unvaccinated have had a previous infection of Covid-19.

A person is selected at random. Given that the person has had a Covid-19 infection determine the probability that this person has been vaccinated …

(a) by using a tree diagram. (6)

(b) by using a two-way (contingency) table. (4)

1.2 The table below summarises a survey of the number of pets per household in a particular town.

Number of pets (x)	0	1	2	3 or more
Percentage of households	18	33	36	13

(a) Write down the probability that a household has at least two pets. (1)

(b) A random sample of ten households is selected from this town. Determine the probability that the sample will contain more than eight households with at least two pets. (6)

(c) A new random sample of n households is selected from this town. The probability that this new sample contains at least one household with more than two pets is greater than 70%. Determine the smallest value of n. (7)

1.3 The discrete random variable X has probability distribution $X \sim B\left(160\,;\frac{1}{8}\right)$.

Use a suitable approximation to find $P(X < 25)$. (8)

[32]

QUESTION 2

The random variable X follows a normal distribution with mean 45 and standard deviation 2.

2.1 Find $P(X < 41 \text{ or } X > 47)$. (8)

2.2 Calculate the 90$^{\text{th}}$ percentile of X. (6)

[14]

QUESTION 3

3.1 A random sample of 30 independent observations of a normally distributed random variable X is taken from a population. A test statistic of $Z = 2,4$ is calculated. It is thought that the population mean is 27. Write a suitable null and alternate hypotheses, and then carry out a two-tailed significance test for the mean at the 1% level. (6)

3.2 A random sample of 80 people from a political party, ABC, were asked whether they were planning to vote in the next election; k people said they were planning to vote. An α% confidence interval for the population proportion of people from ABC who were planning to vote for this party was $(0,468\ ;\ 0,682)$.

(a) Show that $k = 46$. (3)

(b) Hence, find the value of α to the nearest percentage. (7)

It is known that 59,3% of the population of ABC voted in the last election.

(c) If we assume that everyone who planned to vote did vote, is there evidence to say that the proportion of people from ABC who voted, has changed? Give a reason for your answer. (2)

[18]

QUESTION 4

4.1 The probability distribution of a discrete random variable X is given by:

x	1	2	3
$P(X = x)$	0,3 – m	2 m	0,7 – m

(a) Determine $E[X]$. (3)

(b) Find the range of possible values of the constant m. (3)

(c) Given that $\text{var}(X) = 0,72$, find the value of m. (4)

4.2 The lifetime of a certain type of battery, in **tens of hours**, has a probability density function

$$f(x)=\begin{cases}\frac{1}{32}x & 0\le x\le 4\\ \frac{1}{8} & 4<x\le 10\\ 0 & \text{otherwise}\end{cases}$$

(a) Sketch $f(x)$. (6)

(b) Hence using the sketch, show that $P(X>3)=\frac{55}{64}$. (5)

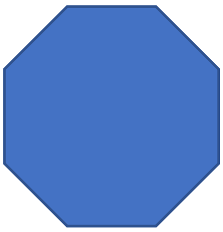
Two batteries are required for a torch to work. The torch will only work if both batteries still have energy.

(c) Vince fits two new batteries into the torch. Determine the probability that the torch will stop working within the next 30 hours. (5)

[26]

QUESTION 5

5.1 The chairperson and vice-chairperson of a committee host a board meeting with six other board members. They all sit around an octagonal table.



Find the number of ways that the seating can be arranged if the chairperson and vice-chairperson sit next to each other. (4)

5.2 Two events X and Y are such that $P(X|Y)=P(X)=P(Y)=P(X'\cap Y')$.
Find $P(X)$. (6)

[10]

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

Tesh takes a loan of R550 000 now, with interest calculated at 8% per annum compounded monthly. The loan is repaid by means of equal monthly payments of R6 000 with the first payment made after 6 months.

1.1 Determine the number of payments that will be made. (7)

1.2 Calculate the value of the final payment. (7)

1.3 Calculate the outstanding balance in 2 years' time and 3 years' time respectively. (6)

1.4 Hence, calculate the interest paid in the 3rd year. (5)

[25]

QUESTION 2

A company wishes to upgrade their IT network in **7 years**' time.

- The financial director calculates that a monthly payment of R30 000 will need to be paid into a sinking fund with an interest rate of 10% per annum, compounded monthly, starting immediately and finishing in 4 years' time.
- A recent quote on the network is R1 750 000 which would rise with inflation at 6% per annum.
- The value of the current equipment is estimated to be Rx and this is depreciating at 20% per annum on a reducing balance.
- The current equipment will be sold in 7 years' time to reduce the cost of the new equipment.

2.1 Determine the value of the sinking fund after 7 years. (7)

2.2 Hence, show that the value of x is R934 874,69. (5)

2.3 The company decides to restructure its monthly payments. Instead, they will make equal monthly payments of R*y* starting immediately and finishing in 7 years' time. In addition, an investor will add a semi-annual payment of R50 000 starting in 6 months' time and finishing in 7 years' time. Assume a target amount of R2 450 000 in 7 years' time.

(a) Convert 10% per annum, compounded monthly, to an equivalent semi-annual rate. (4)

(b) It is given that y can be calculated using the formula:

$$A = \frac{y\left[(1+i)^n - 1\right]}{i} + \frac{q\left[(1+j)^m - 1\right]}{j}$$

Write down, or calculate, the values of A, m, n, q, i and j respectively. (6)

(c) Hence, or otherwise, determine the value of y. (2)

[24]

QUESTION 3

A population of mice is allowed to grow in an area where there are no natural predators. The average lifespan of a mouse is one year. There are 6 litters of 4 pups produced per year with 55% being female. About 80% of the pups survive to maturity. Some mice are killed through mouse traps and mouse repellent.

3.1 Explain why the reproductive cycle should be considered to be 2 months. (2)

3.2 Write down the death rate per 2-month cycle. (1)

3.3 Calculate the intrinsic growth rate per cycle as a percentage, correct to 4 decimal places. (5)

3.4 If there are initially 30 mice, determine the minimum number of mice that must be culled per cycle to stop the population growing. (5)

[13]

QUESTION 4

Data related to the logistic progression of a virus over a 10-month period is given in the table below:

Date	Total infections (P) in 1000's	$\frac{\Delta P}{P}$
1 March 2020	0	
1 April 2020	1,386	2,299
1 May 2020	6,372	2,471
1 June 2020	32,878	2,474
1 July 2020	169,075	x
1 August 2020	496,163	0,467
1 September 2020	632,054	0,144
1 October 2020	y	0,072
1 November 2020	729,835	0,072
1 December 2020	783,270	

4.1 Show by calculation that x = 1,370, correct to three decimal places. (3)

4.2 Calculate y correct to three decimal places. (3)

4.3 Explain how to test if the data can be modelled logistically. Give a rough sketch of the function involved. (You are not required to undertake the mathematics of the test.) (4)

4.4 Given that $\frac{\Delta P}{P} = -0,00339P + 2,335$, determine the carrying capacity and the intrinsic growth rate of the virus. (4)

[14]

QUESTION 5

THIS QUESTION IS MULTIPLE CHOICE. YOU MAY ALSO SHOW WORKING TO GAIN PARTIAL CREDIT.

Refer to the Lotka-Volterra (Predator-Prey) model on the formula sheet, and select the correct option in each of the following:

5.1 Which of the following best describes the term $aR_n\left(1-\frac{R_n}{K}\right)$?

A		The growth in the prey population in each cycle
B		The death rate of the predators
C		The intrinsic growth rate of the prey
D		The number of prey in each litter
E		The equilibrium number of predators

(3)

5.2 The attack rate of foxes on rabbits is 0,04. There are 10 fox kits added to the population in each monthly cycle. The parameter f = 0,0312. The initial number of foxes is 40. An estimate of the initial number of rabbits is:

A		1 000
B		1 600
C		200
D		150
E		100

(3)

5.3 For a particular predator-prey system with a life cycle in months, it is given that $F_{n+1} = 0{,}98F_n + 0{,}00296R_nF_n$.

We can deduce that the lifespan of the predator is:

A		24 months
B		30 months
C		36 months
D		48 months
E		50 months

(3)

5.4 Which of the following is certainly true of the predator-prey model?

A ☐ The initial number of prey is always the same as the carrying capacity.

B ☐ The higher the carrying capacity, the higher the equilibrium number of prey.

C ☐ The equilibrium number of predator is not affected by the intrinsic growth rate of the prey.

D ☐ When the predator numbers are decreasing at the most rapid rate, the number of prey starts to increase.

E ☐ The reproductive growth rate of the predator population depends on its life expectancy.

(3)
[12]

QUESTION 6

A loan of x rand is to be repaid by monthly instalments of *m* rand, starting one month after the granting of the loan, with interest calculated at 12% per annum, compounded monthly. After the first month the outstanding balance is R1 197 000. After the third month, the outstanding balance is R1 190 909,70.

6.1 Write down a recursive formula for the monthly outstanding balance, P_{n+1}. (4)

6.2 Making use of this formula, calculate m. (8)
[12]

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1

1.1 $2x+3y-5z=7$, $-2x+3z=3$ and $-2y+3z=20$ are equations in a 3-dimensional plane. A student places the equations in an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ -2 & 0 & 3 & 3 \\ 0 & -2 & 3 & 20 \end{array}\right)$$

(a) Apply $R_2 + R_1$ to row 2. (2)

(b) Continue the process to solve for *x, y* and *z*, using matrix algebra. (7)

1.2 A point B (–1, 4) is transformed by the matrix $\begin{pmatrix} x & -1 \\ -2 & 3 \end{pmatrix}$, followed by a further transformation of $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

(a) Work out the single matrix of the combined transformation in terms of *x*. (4)

(b) Given *x* = 1, determine the co-ordinates of B', the transformed point. (3)

[16]

QUESTION 2

2.1 $A=\begin{pmatrix} 3 & 8 & 2 \\ 1 & 1 & 1 \\ 5 & 6 & x \end{pmatrix}$, $B=\begin{pmatrix} 1 & 28 & -x-1 \\ 0 & -x & 1 \\ -1 & -2y & x \end{pmatrix}$, if $B=A^{-1}$, using matrix algebra find:

(a) The determinant of matrix *A* in terms of *x*. (4)

(b) The solutions of *x* and *y*. (8)

2.2 Given that $i^2 = -1$. Solve for *x* and *y* if:

$$\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & 1 \end{vmatrix} = 2x+3yi$$ (5)

[17]

QUESTION 3

3.1 (a) A triangle with coordinates, A(1; 1), B(1; 3) and C(2; 1), is transformed to form the matrix: $A'(-1; 1)$, $B'(-3; 1)$ and $C'(-1; 2)$. Find the transformation matrix. (6)

(b) Describe the transformation in words. (3)

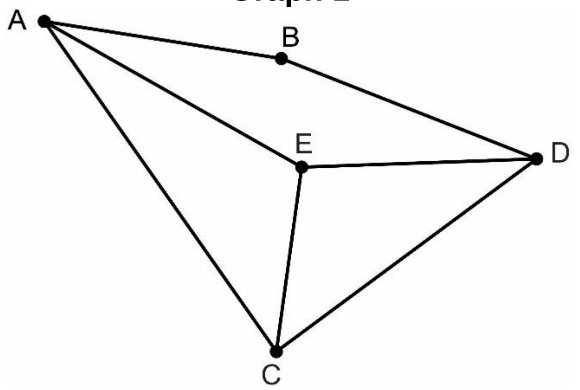
3.2 Let $M = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 2 \end{pmatrix}$. M is enlarged by a factor of 4, and then rotated anticlockwise, by an angle of 30°. Calculate the co-ordinates of M', the image of M. (7)

[16]

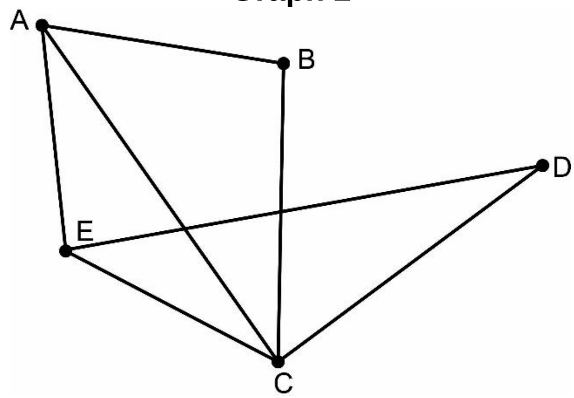
QUESTION 4

4.1

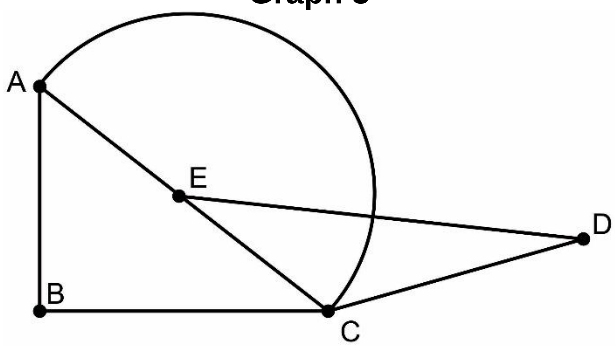
Graph 1



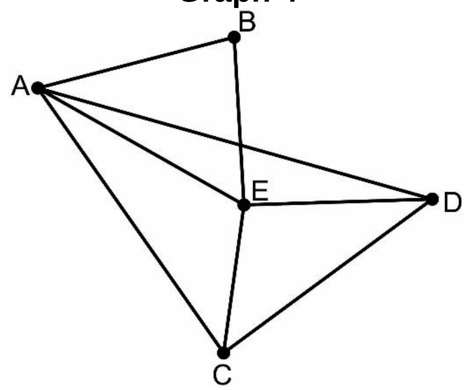
Graph 2



Graph 3



Graph 4

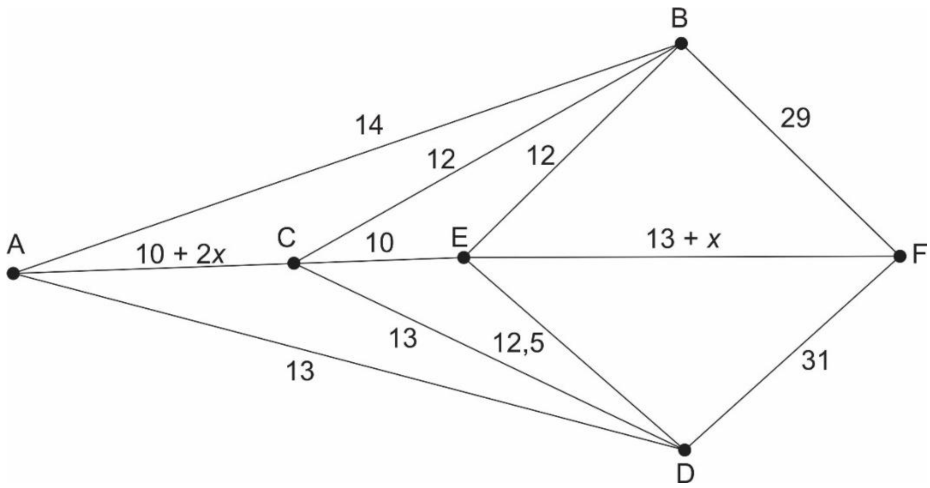


(a) Which of the graphs above are isomorphic? (4)

(b) Draw the compliment of Graph 1 and of Graph 3. (6)

(c) Are the graphs you drew in Question 4.1 (b) isomorphic? (2)

4.2 The diagram below shows the 6 locations in a courier receiving hub. The weights of the edges represent the time taken to transport the package received, where $x > 0$. A package must start at A, and traverse every edge to fulfil the bureaucratic demands of the receiving hub, before returning to A.



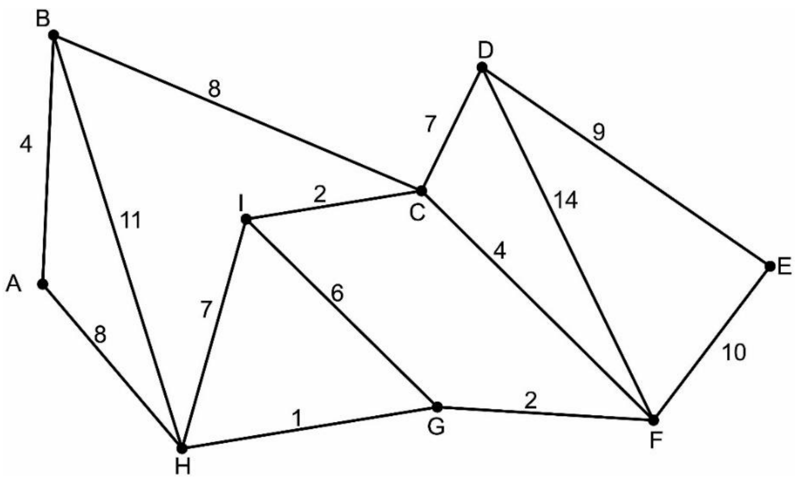
(a) Explain why the path A to F should be repeated. (2)

(b) The route A-C-E is the **second** shortest route, connecting A to E, find the range of possible values for x. (6)

(c) Write a simplified expression, in terms of x, for the minimum time for the package to traverse every edge starting at and returning to A. Write down a possible route. (8)

[28]

QUESTION 5



5.1 Use Dijkstra's algorithm to find the shortest path from A to E. Show clear evidence of your working. Be sure to state your final route, as well as its length. (10)

5.2 Use Prim's algorithm to find the minimum spanning tree, starting at vertex A. Clearly state the order in which you choose the edges, as well as the weight of the tree. (7)

[17]

QUESTION 6

A past Further Studies in Mathematics student is attempting to prove a mathematical statement using mathematical induction. However, they have never used induction with matrices.

Prove by mathematical induction that: $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$ $n \in \mathbb{N}$.

Their attempt is as follows,

For $n = 1$, $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1-2 \\ 0 & 2 \end{pmatrix}$,

$\therefore$ true for $n = 1$.

Assume true for $n = k$, $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix}$.

Complete the process for the student by proving true for $n = k + 1$, $n \in \mathbb{N}$.

[6]

Total for Module 4: 100 marks