GRADE 12 EXAMINATION
NOVEMBER 2021

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

EXAMINATION NUMBER

Time: 2 hours   200 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1.   This question paper consists of 32 pages and an Information Booklet of 4 pages (i–iv). Please check that your question paper is complete.

2.   Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

3.   All necessary calculations must be clearly shown and writing must be legible.

4.   Diagrams have not been drawn to scale.

5.   Round off your answers to 2 decimal digits, unless otherwise indicated.

6.   THREE blank pages (pages 30 to 32) are included at the end of the question paper. If you run out of space for a question, use these pages. Clearly indicate the question number of your answer should you use this extra space.

QUESTION 1

1.1   Solve, for $x \in \mathbb{R}$:

   (a) $2e^x - 7 + 6e^{-x} = 0$, expressing your answers in exact form using logs.

(8)

   (b) $|2x + 3| = 5x - 2$

(6)

(c) $\dfrac{\left(-x^2-5\right)\left(x^2-16\right)}{\left|x+3\right|\left(x+2\right)} \geq 0$

(8)

1.2   Consider the function $f(x) = x^4 - 3x^3 - 5x^2 + 29x - 30$

(a)   Given that $x = 2 - i$ is a root to the equation $f(x) = 0$, write $f(x)$ as a product of two trinomials.

(4)

(b)   Hence, solve $f(x) = 0$ in $\mathbb{C}$.

(2)

1.3　　Kwande defined a new type of complex number called a **Kwande number**.
　　　　It has the property that $\text{Im}(z) = 2\text{Re}(z)$.

　　　　In other words, a **Kwande number**, z is of the form: $z = a + 2ai$.

　　　　Prove that for all **Kwande numbers** $\dfrac{z}{z^*} = -\dfrac{3}{5} + \dfrac{4}{5}i$

<div align="right">

(5)
[33]

</div>

QUESTION 2

The number of people, n, in a school with population P who have heard a rumour can be modelled by the following function:

$$n = P - Pe^{-0.14t}$$

where t is the time (in days) that have elapsed since the rumour began.

(a)   Make t the subject of the formula.

(4)

(b)   Hence, determine how many days, to the nearest day, it will take for at least 750 people in a school of 1200 to have heard the rumour.

(2)
[6]

QUESTION 3

[8]

By first principles show that the derivative of $f(x) = \dfrac{1}{2+3x}$ is $\dfrac{-3}{(2+3x)^2}$.
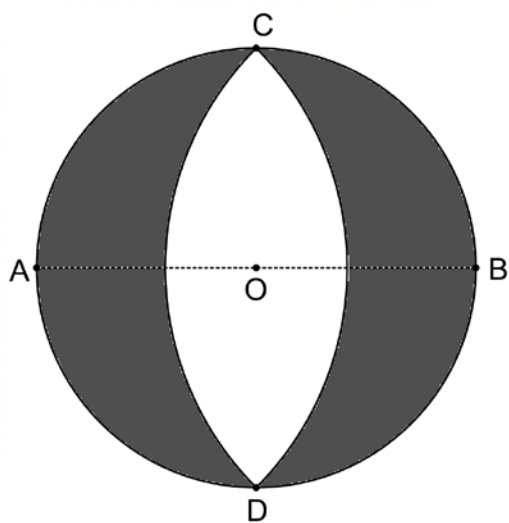
QUESTION 4

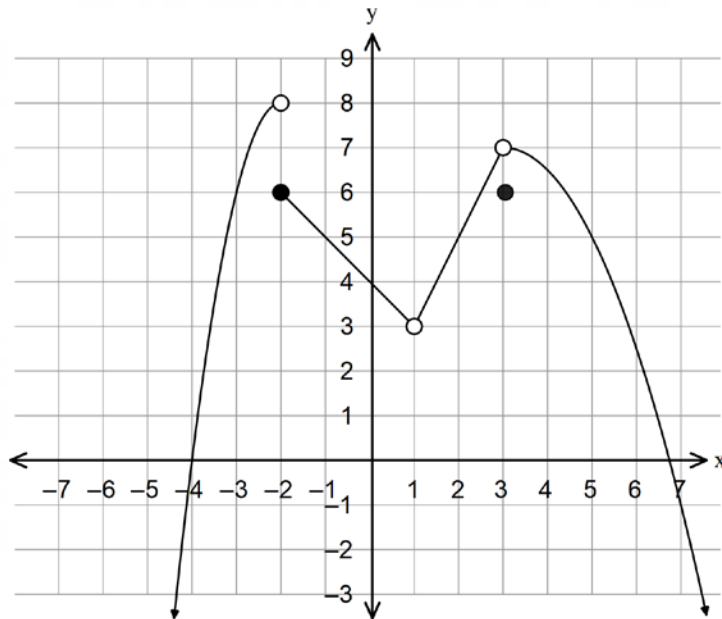Prove that $3^{2n+4} - 2^{2n}$ is a multiple of 5 for all $n \in \mathbb{N}.$

[13]

QUESTION 5

In the diagram below, the circle, centre O, with diameter AB has a radius of 4 cm. Circular arcs are drawn through C and D with A and B as centres.

Determine the shaded area.

[10]

QUESTION 6

6.1   Consider the graph of the function f shown below.

Answer the following questions paying careful attention to the precision of mathematical notation you use:

(a)   Using mathematical notation, justify why f is discontinuous at $x = -2$.

(2)

(b)   Using mathematical notation, justify why f is discontinuous at $x = 1$.

(2)

(c)   Using mathematical notation, justify why *f* is discontinuous at $x = 3$.

(2)

(d)   What is the nature of the discontinuity at $x = -2$?

(1)

(e)   What is the nature of the discontinuity at $x = 3$?

(1)

6.2   Explain why $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

(4)

6.3   Consider the function g, defined below.

$$g(x) = \begin{cases} -0{,}5x^2 + 2x + 3 & x \leq 2 \\ px^2 + qx + 13 & x > 2 \end{cases}$$

Using appropriate mathematical notation, determine the rational values of p and q if g is differentiable at $x = 2$.

(8)
[20]

QUESTION 7

The curve $2x^2 - xy = x^3 + y^2$ has two tangents where the x-coordinate of the point of contact is 2. Find the equation of the tangent with a positive gradient.

[10]
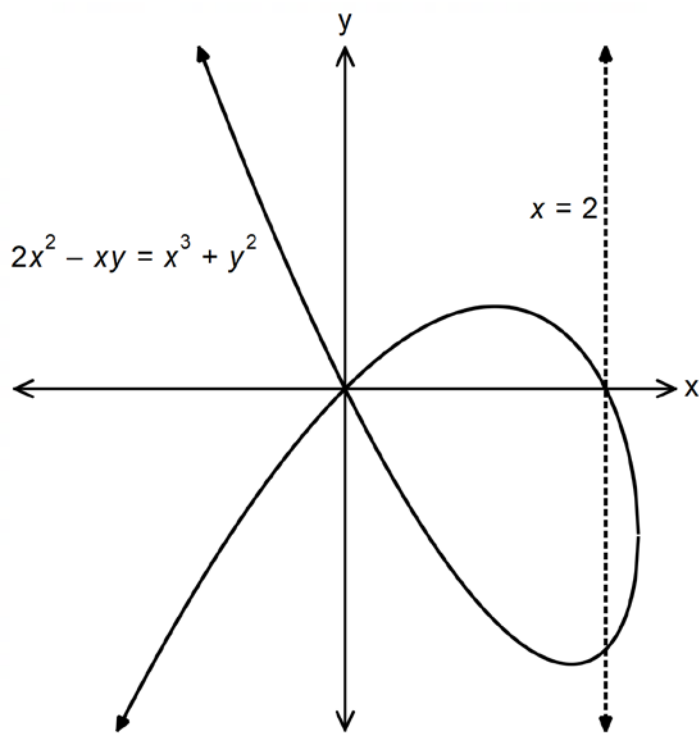
QUESTION 8

8.1   For each of the given functions, determine integral value(s) of *a* if:

(a) $f(x) = \dfrac{ax^2 + 2x + 3}{-\dfrac{1}{2}x^2 + 3x + 4}$ has an asymptote $y = 2$

(2)

(b) $f(x) = \dfrac{x^2 + 4}{x^2 - 4x + a}$ has asymptotes $x = 1$ and $x = 3$

(2)

(c) $f(x) = \dfrac{2x^2 + 2x + 3}{x + a}$ has an asymptote $y = 2x - 4$

(6)

(d) $f(x) = \dfrac{x^2 + 2x + 3}{x^a + 3x + 4}$ has an asymptote $y = 0$
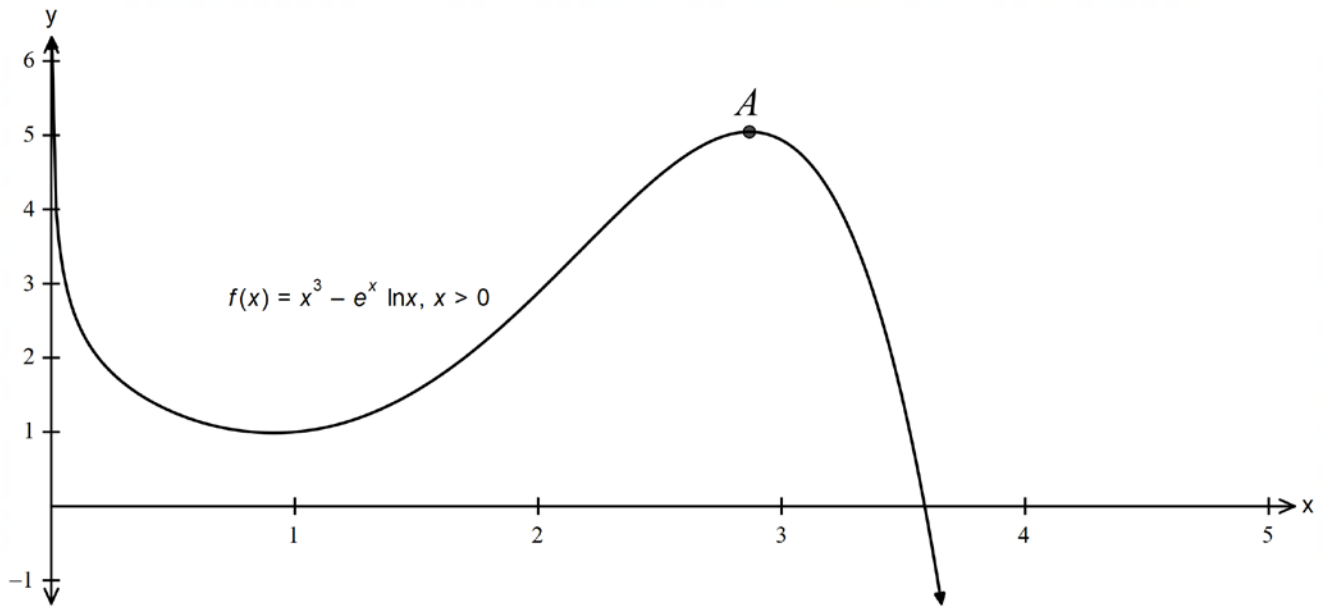
(2)

8.2   Determine the *x*-coordinate(s) of the stationary point(s) of
$$f(x) = \frac{2x^2 - 7x + 1}{2x + 3}$$

(6)
[18]

QUESTION 9

A portion of the function $f(x) = x^3 - e^x \ln x, \ x > 0$ is shown.

9.1   Show that the equation below would be the one you would have to solve to find the x-coordinate of the local maximum at point A.

$$3x^2 - e^x\left(\ln x + x^{-1}\right) = 0$$

(6)

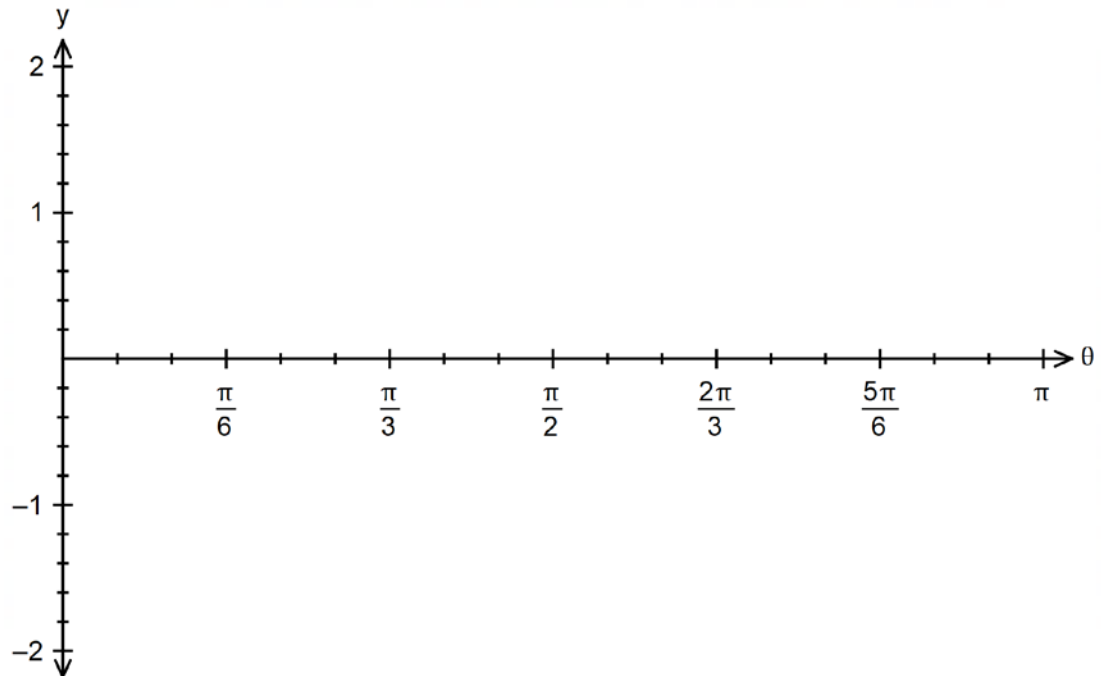9.2   Use Newton-Raphson iteration to find, to five decimal places, the *x*-coordinate of A.

- show the iterative formula you use.
- use $x_0 = 3$ as the first approximation.
- show the value for x_1 to five decimal places.

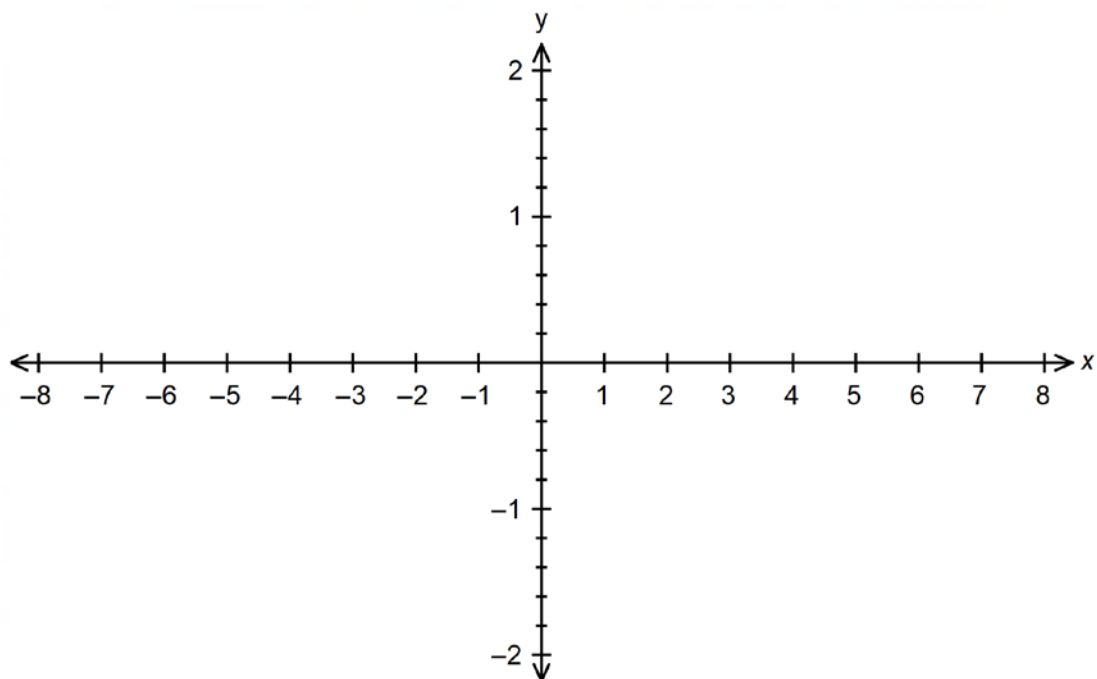(8)
[14]

QUESTION 10

10.1   Sketch the following functions, indicating the *x*-intercepts:
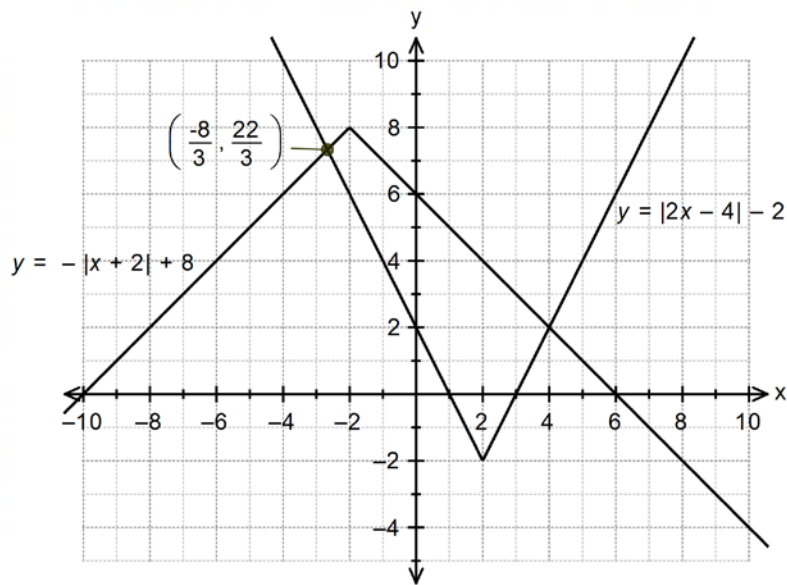
(a) $y = |\sin 2\theta|$ for $\theta \in [0; \pi]$

(4)

(b) $y = \ln|x|$

(4)

10.2   Use the graphs on the scaled axes below, or otherwise, to solve the given inequalities:

$\left(\dfrac{-8}{3}, \dfrac{22}{3}\right)$

$y = -|x + 2| + 8$

$y = |2x - 4| - 2$
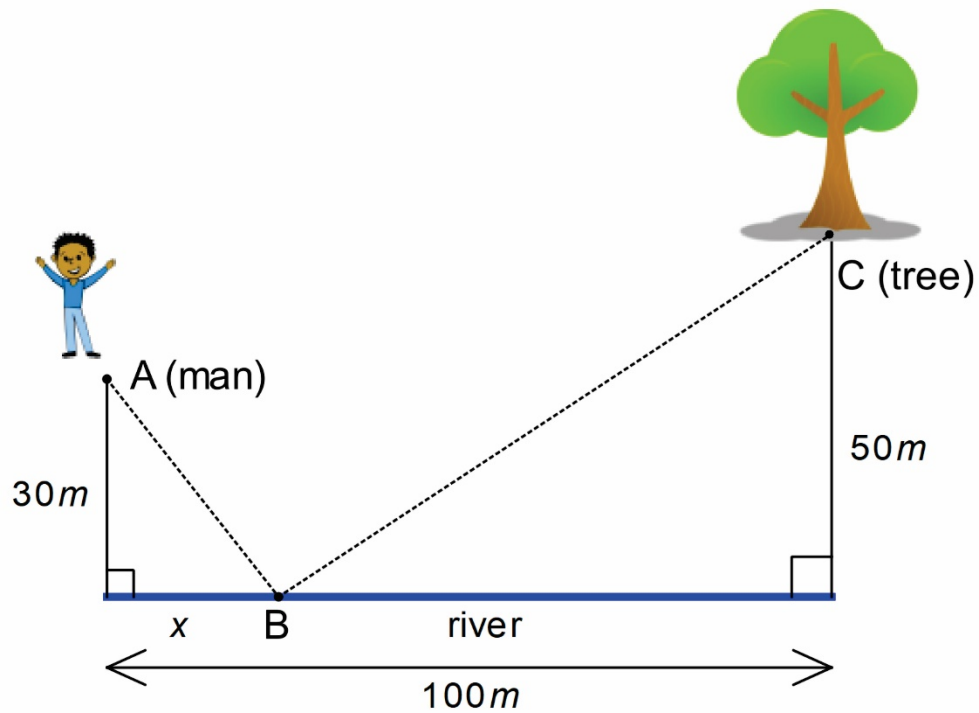
(a) $|2x - 4| \le 8$

(4)

(b) $2|x - 2| + |x + 2| \le 10$

(6)
[18]

QUESTION 11

11.1 A man is standing 30 m away from a straight river. 100 m downstream there is a tree which is 50 m from the riverbank. He wishes to walk to the river to drink and then to the tree to rest in the shade. He walks in straight lines as depicted by the dotted lines in the diagram.


(a) Show that the distance he will walk is given by the expression:

$$d = \sqrt{x^2 + 900} + \sqrt{x^2 - 200x + 12\,500}$$

(4)

(b)   Hence, determine the value of x which will minimise the distance d.

(8)

11.2 Determine the area labelled A below. It is bounded above by the curve $f(x) = \dfrac{8}{x-1} + 1$, below by the curve $g(x) = \dfrac{12}{x+4}$, to the left by the line $x = 3$ and to the right by the line $x = 9$. You should show the expression involving the integrals which you use to calculate your answer.

(6)
[18]
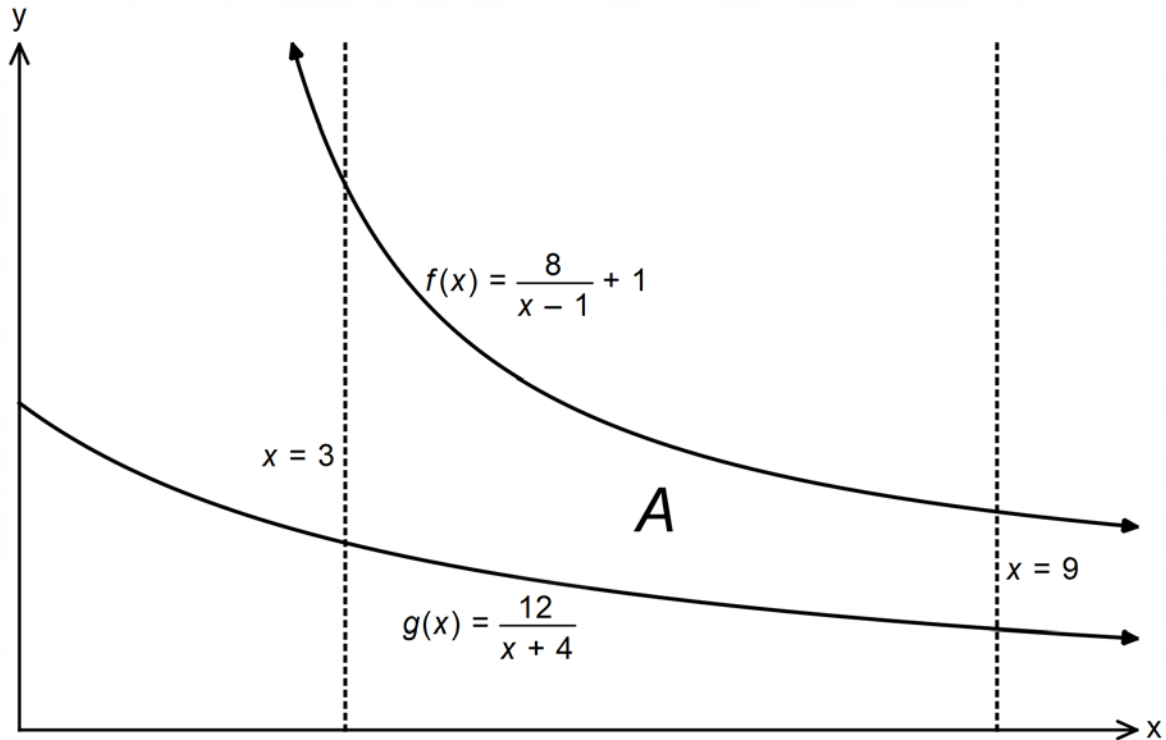
QUESTION 12

12.1   Determine the following:

(a) $\int \sin 5x \cos 4x \, dx$

(6)

(b) $\int xe^x \, dx$

(4)

(c) $\displaystyle\int e^{\tan 2x} \sec^2 2x \, dx$

(4)

(d) $\int \dfrac{x^3 - 3}{x^2 - 1}\, dx$
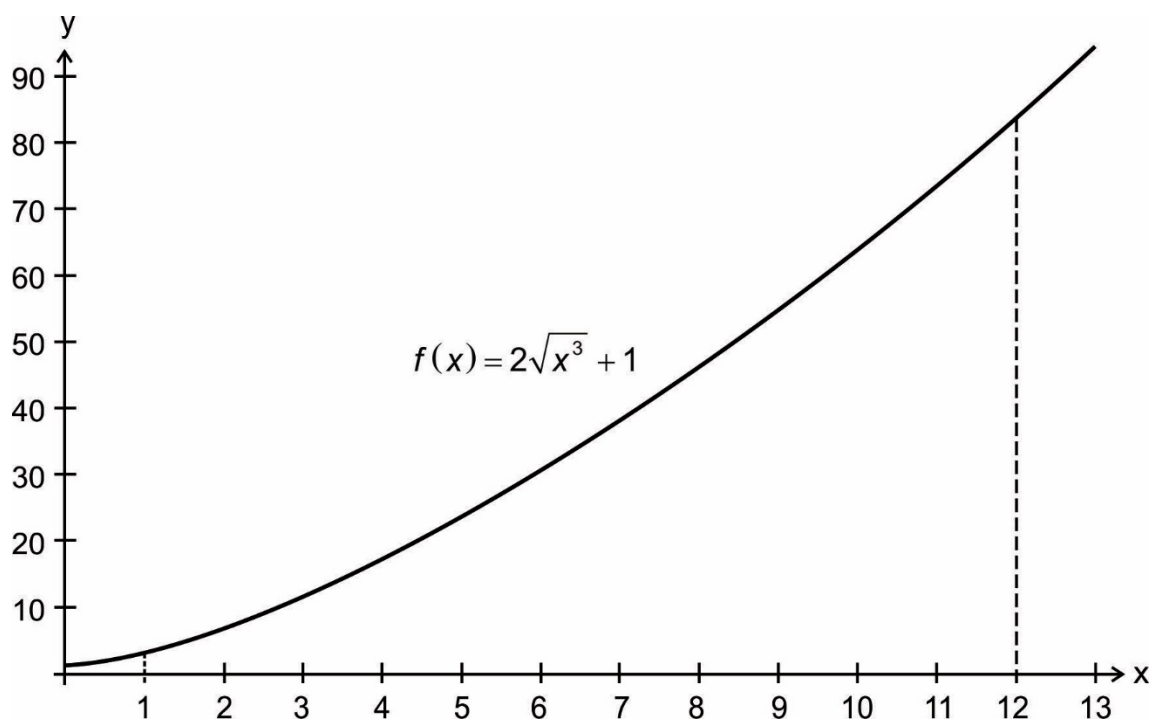
(11)

12.2   The arc length of a function $f(x)$ from $x=a$ to $x=b$ is given by the formula:

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\ dx$$

Given the function $f(x) = 2\sqrt{x^3} + 1$

Use this formula to determine the arc length of the $f(x)$ between $x=1$ and $x=12$ as illustrated below.

You must show the integral which you use to calculate your answer.

(7)
[32]

Total: 200 marks

ADDITIONAL SPACE TO ANSWER QUESTIONS. REMEMBER TO CLEARLY INDICATE AT THE QUESTION THAT YOU USED THE ADDITIONAL SPACE TO ENSURE THAT ALL THE ANSWERS ARE MARKED.