INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2022

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

MARKING GUIDELINES

Time: 1 hour 100 marks

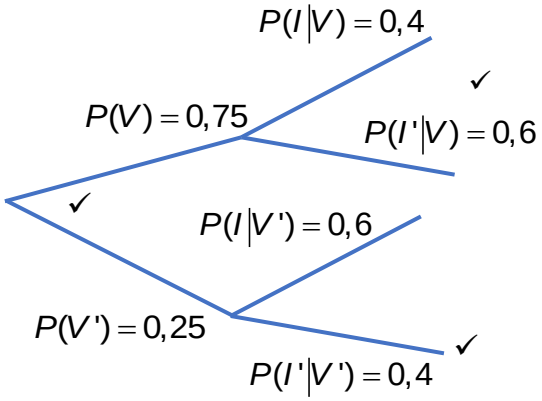
These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

MODULE 2 STATISTICS

QUESTION 1

1.1 (a)



$$P(V|I)=\frac{(0,75)(0,4)}{(0,75)(0,4)+(0,25)(0,6)}=\frac{2}{3}$$ (6)

(b)

	Previous infection	No previous infection	Total
Vaccinated	30%	45%	75%
Not vaccinated	15%	10%	25%
Total	45%	55%	100%

$$P(V|I)=\frac{30\%}{45\%}=\frac{2}{3}$$ (4)

1.2 (a) $P(X\geq 2)=0,49$ (1)

(b) $X\sim B(10;0,49)$

$$P(X>8)=P(X=9)+P(X=10)$$
$$=\binom{10}{9}(0,49)^9(0,51)+\binom{10}{10}(0,49)^{10}(0,51)^0$$
$$=0,0091$$ (6)

(c) $Y\sim B(n;0,13)$

$$P(Y\geq 1)>0,7$$
$$1-\binom{n}{0}(0,13)^0(0,87)^n>0,7$$
$$(0,87)^n<0,3$$
$$n>\frac{\log 0,3}{\log 0,87}$$
$$n>8,64$$
$$\therefore n=9$$ (7)

1.3 $X \sim B\left(160;\frac{1}{8}\right) \rightarrow X \sim N\left(20;\sqrt{17,5}^2\right)$ ✓ ✓

$P(X < 25) \rightarrow P(X < 24,5)$ ✓

$= P\left(Z < \frac{24,5 - 20}{\sqrt{17,5}}\right)$ ✓

$= P(Z < 1,08)$ ✓

$= 0,5$ ✓ $+ 0,3599$ ✓

$= 0,8599$ ✓ (8)

[32]

QUESTION 2

2.1 $X \sim N(45;2^2)$

$P(X < 41 \text{ or } X > 47) = 1 - P(41 < X < 47)$ ✓

$= 1 - P\left(\frac{41-45}{2} < Z < \frac{47-45}{2}\right)$ ✓ ✓

$= 1 - P(-2 < Z < 1)$ ✓

$= 1 - (0,4772 + 0,3413)$ ✓ ✓ ✓

$= 0,1815$ ✓ (8)

2.2 $P(X < a) = 0,9$ ✓

$1,28 = \frac{a - 45}{2}$ ✓ ✓ ✓ ✓

$a = 47,56$ ✓ (6)

[14]

QUESTION 3

3.1 $H_0 : \mu = 27$ ✓
$H_1 : \mu \neq 27$ ✓

Rejection region: reject H_0 if $|Z| > 2{,}58$ ✓✓

Test statistic: $Z = 2{,}4$

Conclusion: since $Z < 2{,}58$ fail ✓ to reject H_0 at the 1% level of significance and suggest ✓ insufficient evidence to support the claim. (6)

3.2 (a) $\frac{k}{80}$ ✓ $= \frac{0{,}468 + 0{,}682}{2}$ ✓

$k = 80\left(\frac{23}{40}\right)$ ✓

$k = 46$ (3)

(b) $0{,}575 + Z\sqrt{\frac{(0{,}575)(0{,}425)}{80}}$ ✓ $= 0{,}682$ ✓

$Z = 1{,}94$ ✓✓

$\therefore P(-1{,}94 < Z < 1{,}94) = 2(0{,}4738)$ ✓

$= 0{,}9476$

$\therefore \alpha = 95\%$ ✓ (7)

(c) 59,3% ✓ is in the confidence interval hence there is evidence to say that the proportion of people who voted has not changed. ✓ (2)

[18]

QUESTION 4

4.1 (a) $E[X] = 1(0,3-m) + 2(2m) + 3(0,7-m)$ ✓

$= 0,3 - m + 4m + 2,1 - 3m$ ✓

$= 2,4$ ✓ (3)

(b) $0,3 - m \geq 0$ $\quad 2m \geq 0$ $\quad 0,7 - m \geq 0$

$m \leq 0,3$ ✓ $\quad m \geq 0$ $\quad m \leq 0,7$ ✓

$\therefore 0 \leq m \leq 0,3$ ✓ (3)

(c) $\text{var}(X) = E[X^2] - (E[x])^2$

$0,72 = E[X^2] - (2,4)^2$ ✓

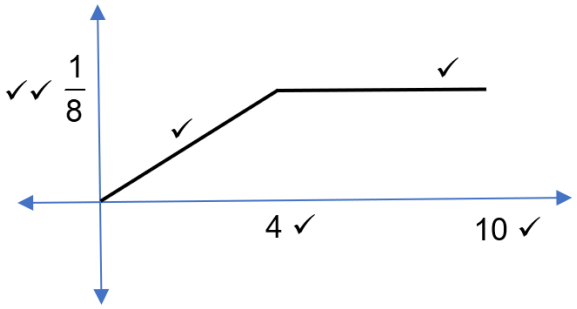
$E[X^2] = 6,48$ ✓

$0,3 - m + 4(2m) + 9(0,7 - m) = 6,48$ ✓

$-2m = -0,12$

$m = 0,06$ ✓ (4)

4.2 (a)



(6)

(b) $P(X > 3) = 1 - \frac{1}{2}(3)\left(\frac{3}{32}\right) = \frac{55}{64}$ ✓✓ ✓ ✓ ✓ (5)

(c) $1 - \left(\frac{55}{64}\right)^2$ ✓ ✓ ✓ $= 0,2615$ ✓

OR

$\left(\frac{55}{64}\right)\left(\frac{9}{64}\right) \times 2 + \left(\frac{9}{64}\right)^2 = 0,2615$ (5)

[26]

QUESTION 5

5.1 $(7-1)!\times 2 = 1440$ ✓ ✓ ✓ (4)

5.2 Let $P(X) = P(Y) = P(X' \cap Y') = x$

$$\therefore P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) \checkmark$$
$$1 - P(X' \cap Y') \checkmark = P(X) + P(Y) - P(X).P(Y) \checkmark$$
$$1 - x = x + x - x^2 \checkmark$$
$$x^2 - 3x + 1 = 0 \checkmark$$
$$x = 0{,}382 \checkmark \quad \textit{or} \quad x \neq 2{,}618$$

OR

Let $P(X|Y) = P(X) = P(Y) = P(X' \cap Y') = k$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$
$$\therefore P(X \cup Y) = 2k - P(X \cap Y)$$
$$P(X' \cap Y') = 1 - P(X \cup Y)$$
$$\therefore k = 1 - 2k + P(X \cap Y)$$
$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$
$$\therefore P(X \cap Y) = k^2$$
$$\therefore k = 1 - 2k + k^2$$
$$k^2 - 3k + 1 = 0$$
$$k = 0{,}382$$

(6)
[10]

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

1.1 $$550\,000\left(1+\frac{0{,}08}{12}\right)^{5} = \frac{6000\left[1-\left(1+\frac{0{,}08}{12}\right)^{-n}\right]}{\frac{0{,}08}{12}}$$ ✓ ✓ ✓ ✓ ✓

n = 150,34

∴ 151 payments ✓ (7)

1.2 O.B. after 150 payments =

$$OB(2\,years) = 550\,000\left(1+\frac{0{,}08}{12}\right)^{155} - \frac{6000\left[\left(1+\frac{0{,}08}{12}\right)^{150}-1\right]}{\frac{0{,}08}{12}}:$$ ✓ ✓ ✓ ✓

$$= R2090{,}19$$

∴ Final payment = $2090{,}19\left(1+\frac{0{,}08}{12}\right)$ ✓✓

= R2104,12 ✓ (7)

1.3 $$OB(2\,years) = 550\,000\left(1+\frac{0{,}08}{12}\right)^{24} - \frac{6000\left[\left(1+\frac{0{,}08}{12}\right)^{18}-1\right]}{\frac{0{,}08}{12}}$$ ✓ ✓

$$= R530\,745{,}22$$ ✓

$$OB(3\,years) = 550\,000\left(1+\frac{0{,}08}{12}\right)^{36} - \frac{6000\left[\left(1+\frac{0{,}08}{12}\right)^{30}-1\right]}{\frac{0{,}08}{12}}$$ ✓

$$= R500097{,}25$$ ✓

(6)

1.4 ✓ ✓ ✓ ✓

$12 \times 6000 - (530745{,}22 - 500097{,}25)$

$= R41\,352{,}03$ ✓ (5)

[25]

QUESTION 2

2.1 Amount required:

$$\frac{30\,000\checkmark\left[\left(1+\frac{0,1}{12}\right)^{49}\checkmark-1\right]\bullet\checkmark\left(1+\frac{0,1}{12}\right)^{36}\checkmark\checkmark}{\frac{0,1}{12}}$$

$$= R2435295,52 \quad \checkmark$$

(7)

2.2 $1\,750\,000\checkmark(1+0,06)^{7}\checkmark-x\checkmark(1-0,2)^{7}\checkmark=2\,435\,295,52$ [spacing for millions] (5)

2.3 (a) $\left(1+\frac{r}{2}\right)^{2}\checkmark=\left(1+\frac{0,1}{12}\right)^{12}\checkmark$

r = 0,1021066 … ✓ (4)

(b) $$2\,450\,000=\frac{y\left[\left(1+\frac{0,1}{12}\right)^{85}-1\right]}{\frac{0,1}{12}}+\frac{50\,000\left[\left(1+\frac{0,1021}{2}\right)^{14}-1\right]}{\frac{0,1021}{2}}$$

A = 2 450 000 ✓

q = 50 000 ✓

m = 85 ✓

n = 14 ✓

i = 0,00833 ✓

j = 0,05105 ✓

(6)

(c) $y = R11\,897,32$ ✓✓ (2)

[24]

QUESTION 3

3.1 6 litters per 12 months ✓

$\frac{12}{6}$ = 2 months ✓ (2)

3.2 1/6 or 16,7% ✓ (1)

3.3 r = b – d ✓

= 0,55 ✓ × 0,8 ✓ × 4 – 1/6 ✓

= 1.5933 ✓ (5)

3.4 $P_{n+1} = 2,5933Pn$ $P_o = 30$

P_o ✓ $= 2.5933P_o$ ✓ $- x$ ✓

$x = 1,5933 \times 30$ ✓

= 47,799

∴ 48 minimum ✓ (5)

[13]

QUESTION 4

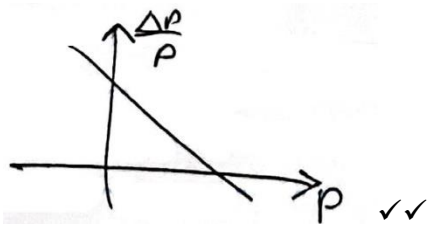
4.1 $x = \frac{496,163 - 32,878}{2 \times 169,075}$ ✓ ✓ ✓

= 1,370 (3)

4.2 $\frac{729,835 - 632,054}{2y}$ ✓ $= 0,072$ ✓

∴ y = 679,035 ✓ (3)

4.3 Show that $\frac{\triangle P}{P}$ plotted against P is a straight line ✓✓



✓✓

(4)

4.4 O = –0,00339K + 2,335 ✓✓

∴ K = 689 ✓

r = 2,335 ✓ (4)

[14]

QUESTION 5

5.1 (A) ✓✓✓ (3)

5.2 $f.b.R_n.F_n = 10$ b = 0,04

$0,0312 \times 0,04 \times R_O \times 40 = 10$

$\therefore R_O = 200$

(C) ✓✓✓ (3)

5.3 $F_{n+1} = F_n + fbR_nF_n - cF_n$

$F_{n+1} = F_n + 0,00296R_nF_n - 0,02F_n$

$Lifespan = \frac{1}{0,02}$

= 50 months

(E) ✓✓✓ (3)

5.4 (D) ✓✓✓ (3)

[12]

QUESTION 6

6.1 $P_{n+1} = P_n\left(1+\frac{0,12}{12}\right) - m$ ✓✓

$P_{n+1} = 1,01P_n - m$ ✓ $P_o = x$ ✓ (4)

6.2 $P_1 = 1,01P_o - m$ ✓

$1\ 197\ 000 = 1,01P_o - m$ ✓

$P_3 = 1,01P_2 - m$ ✓

$P_3 = 1,01(1,01P_1 - m) - m$ ✓

$P_3 = 1,0201P_1 - 1,01m - m$ ✓

$1190\ 909,70 = 1,01^2(1197\ 000) - 2,01m$ ✓✓

m = 15 000✓ (8)

[12]

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1

1.1 (a) $\left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & -2 & 10 \\ 0 & -2 & 3 & 20 \end{array}\right)$ ✓✓

(2)

(b) $3R_3 + 2R_2$ ✓✓ $\left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & -2 & 10 \\ 0 & 0 & 5 & 80 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & -2 & 10 \\ 0 & 0 & 1 & 16 \end{array}\right)$ $Z = 16$ ✓

$R_2 + 2R_3$ ✓✓ $\left(\begin{array}{ccc|c} 2 & 3 & -5 & 7 \\ 0 & 3 & 0 & 42 \\ 0 & 0 & 1 & 16 \end{array}\right)$ $y = 14$ ✓ $x = \frac{45}{2}$ ✓

(7)

1.2 (a) $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x & -1 \\ -2 & 3 \end{pmatrix}$ ✓

$\begin{pmatrix} x+0 & -1+0 \\ 3x+4 & -3-6 \end{pmatrix} = \begin{pmatrix} x & -1 \\ 3x+4 & -9 \end{pmatrix}$ ✓✓✓ (4)

(b) ✓$\begin{pmatrix} 1 & -1 \\ 7 & -9 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -43 \end{pmatrix}$ ✓✓ (3)

[16]

QUESTION 2

2.1 (a) $\det = \checkmark\ 3(x-6)-8(x-5)+2(6-5)\checkmark$

$= 3x - 18 - 8x + 40 + 2$

$= -5x + 24\checkmark\checkmark$ (4)

(b) $\dfrac{1}{-5x+24}\begin{pmatrix} x-6 & -8x+12 & 6 \\ -x+5 & 3x-10 & -1 \\ 1 & 22 & -5 \end{pmatrix}$ ✓✓✓✓

$\dfrac{x-6}{-5x+24} = 1$ ✓✓

$22 = 2y$

$x = 5\checkmark$ $y = 11\checkmark$ (8)

2.2 $12 - 60i + 18i + 60i + 18i^2 + 12i\checkmark\checkmark$

$= -6 + 30i\checkmark$

$\therefore x = -3\ \checkmark\ y = 10\checkmark$ (5)

[17]

QUESTION 3

3.1 (a) $\checkmark\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

$a + b = -1$ ✓

$a + 3b = -3$

$c + 3d = 1$ ✓

$c + d = 1$

$a = 0, b = -1; c = 1$ and $d = 0$ ✓✓✓

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ (6)

(b) Rotation ✓ of 90 ✓ anticlockwise, ✓ about the origin. (3)

3.2 $\checkmark\begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix}\begin{pmatrix} 4 & 12 & 0 \\ 8 & 16 & 8 \end{pmatrix} = \begin{pmatrix} -0.535 & 2.392 & -4 \\ 8.928 & 19.850 & 6.928 \end{pmatrix}$

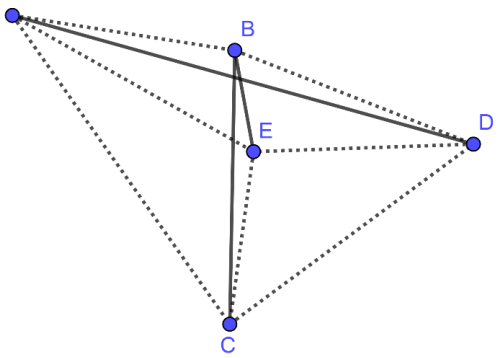
✓ ✓✓ ✓✓✓ (7)

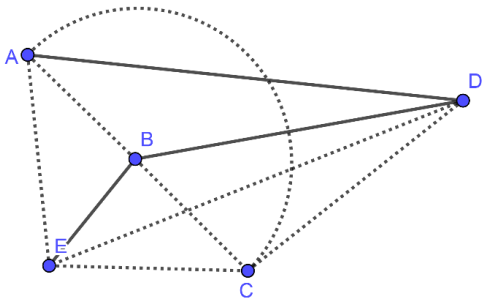
[16]

QUESTION 4

4.1 (a) Graph 2 and 3 ✓✓✓✓ (4)

(b)





✓✓✓ ✓✓✓ (6)

(c) No, they aren't, ✓✓ (2)

4.2 (a) A & F have odd degree edges ✓✓ (2)

(b) ABE = 26, ✓ADE = 25.5 ✓
$25.5 < 20 + 2x < 26$ ✓✓
$2.75 < x < 3$✓✓ (6)

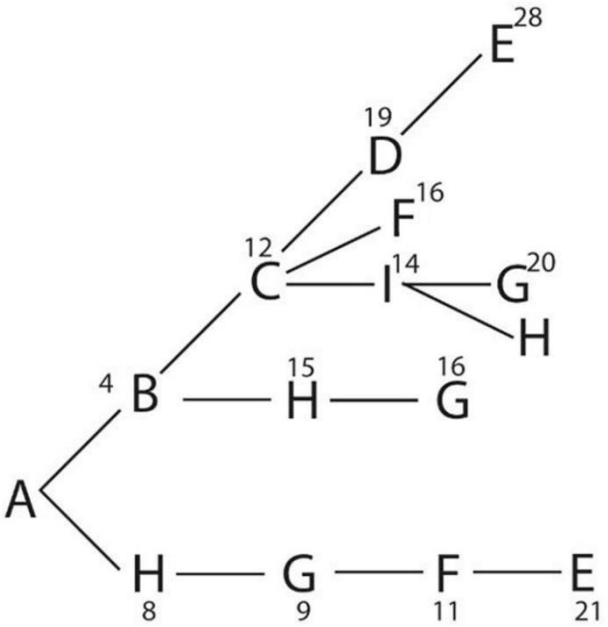
(c) 11 edges starting at A and ending at F, ✓
then add returning to A
Odd vertex A and F,
Shortest path to repeat is FE, ED and DA ✓✓
Total weight 208 + 4x, ✓✓
A-B-C-A-D-C-E-B-F-D ✓✓✓ -E-F -E-D-A (8)

[28]

QUESTION 5

5.1

✓✓



✓✓✓✓✓✓

A H G F E = 21 ✓ ✓ (10)

5.2 AB – 4 ✓
AH – 8 / BC –8✓
HG – 1 / CI – 2
GF – 2 / CF – 4✓
FC – 4 / GF – 2 ✓
CI – 2 / HG – 1
CD – 7 / CD – 7 ✓
DE – 9 / DE – 9 ✓
= 37 ✓ (7)

[17]

QUESTION 6

Let n = k + 1

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \checkmark$$

$$= \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \checkmark\checkmark$$

$$= \begin{pmatrix} 1 & -1+2\left(1-2^k\right) \\ 0 & 2^{k+1} \end{pmatrix} \checkmark$$

$$= \begin{pmatrix} +1 & -1+2-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix} \checkmark$$

$$= \begin{pmatrix} +1 & 1-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix} \checkmark$$

Therefore, true for n = k + 1

By the Principle of Mathematical Induction, true for all $n \in \mathbb{N}$.

[6]

Total for Module 4: 100 marks