Mathematics

CLASS TEXT & STUDY GUIDE

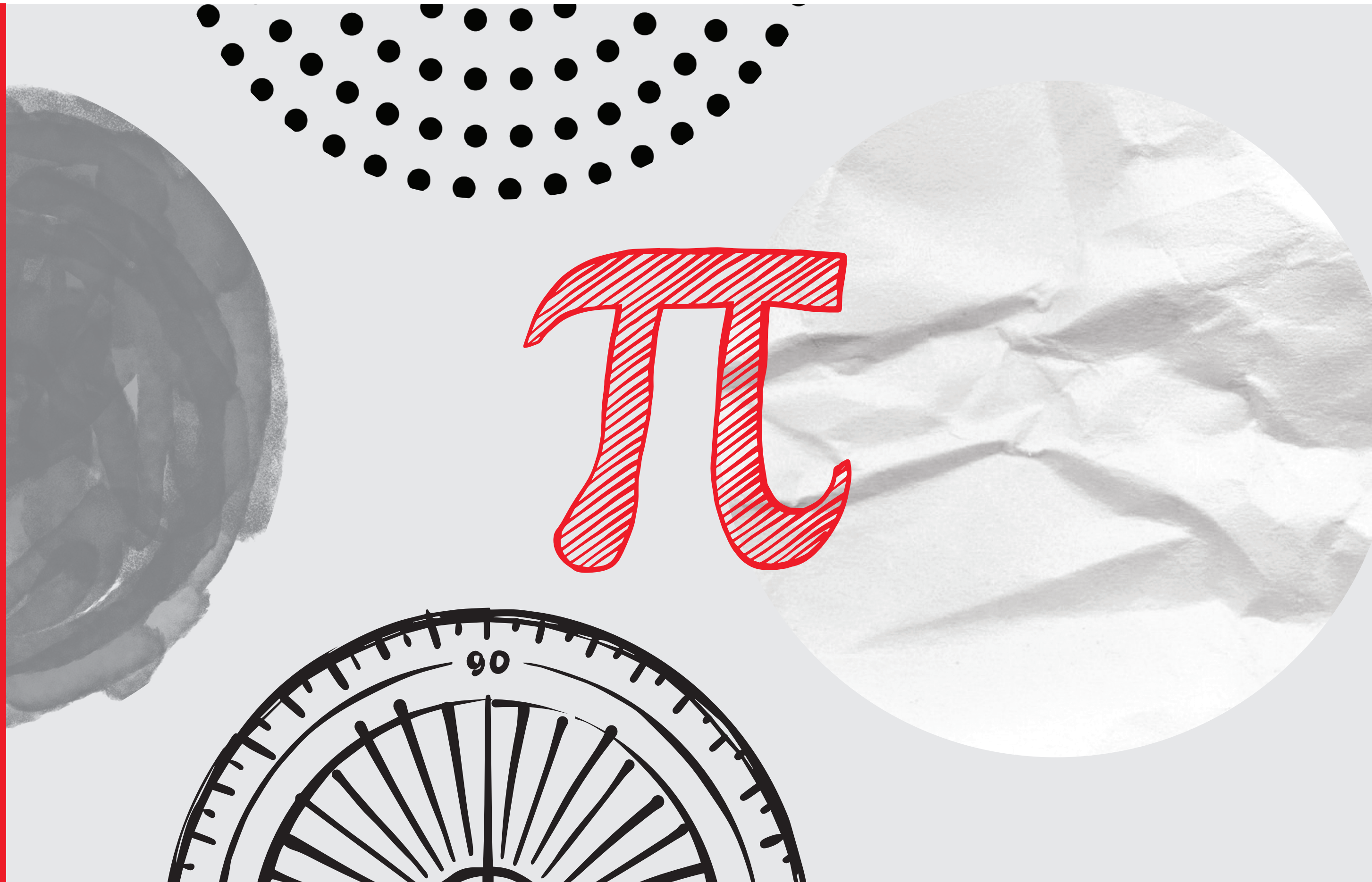
Anne Eadie & Gretel Lampe

GRADE

11

CAPS

3-in-1



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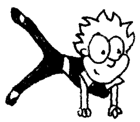
CONTENTS



MODULES

EXAMS

Practise and study the following two exam papers very carefully:

Algebraic Expressions and Equations

It is extremely important to differentiate between expressions and equations. Observe the following examples.

Expressions

- Simplify: $\frac{x}{2} + 3$

$\frac{x}{2} + 3$

$= \frac{x + 6}{2}$ ◄

Do not multiply.

Equations

- Solve for x: $\frac{x}{2} + 3 = 0$

$\frac{x}{2} + 3 = 0$

$\times 2)$ $\therefore x + 6 = 0$

$\therefore x = -6$ . . . *a solution*

Do multiply.

- Multiply: $(x - 2)(x + 5)$

$(x - 2)(x + 5)$ . . . *factors*

$= x^2 - 2x + 5x - 10$

$= x^2 + 3x - 10$ . . . *terms*

↑ Note the **=** signs.

The value of the expression must not change.

- Solve for x: $(x - 2)(x + 5) = 0$

Don't multiply. You need the factors for a zero product:

$(x - 2)(x + 5) = 0$

$\therefore x - 2 = 0$ or $x + 5 = 0$

$\therefore x = 2$ $\therefore x = -5$

↑ Note the $\therefore$ signs.

logic

- Factorise: $x^2 - 9$

$x^2 - 9$ . . . *terms*

$= (x + 3)(x - 3)$ . . . *factors*

↑ Note the equal signs (**=**) down the left.

- Solve for x: $x^2 - 9 = 0$

Method 1: $(x + 3)(x - 3) = 0$

$\therefore x + 3 = 0$ or $x - 3 = 0$

$\therefore x = -3$ $\therefore x = 3$

Method 2:

$x^2 - 9 = 0$ ➡ $x^2 = 9$

$\therefore x = \pm 3$

- Factorise: $-2x^2 + 14x - 24$

$-2x^2 + 14x - 24$ . . . *a trinomial*

$= -2(x^2 - 7x + 12)$

$= \mathbf{-2}(x - 4)(x - 3)$

Keep the value; so, keep the **-2**

- Solve for x: $-2x^2 + 14x - 24 = 0$

$-2x^2 + 14x - 24 = 0$

$\div$ **(-2)** $\therefore x^2 - 7x + 12 = 0$

$\therefore (x - 4)(x - 3) = 0$

$\therefore x = 4$ or $x = 3$

Logic allows you to **DIVIDE** both sides of the equation by **-2**.

- Factorise: $3x^2 - 6x$

$3x^2 - 6x$ . . . *terms*

$=$ **3x**$(x - 2)$ . . . *factors*

The expression was transformed by taking out a **common factor**, and **keeping it**!

- Solve for x: $3x^2 - 6x = 0$

$3x^2 - 6x = 0$

Divide only **by 3**, not $3x$

$\div$ **3)** $\therefore x^2 - 2x = 0$

$\therefore x(x - 2) = 0$

$\therefore x = 0$ **or** $x - 2 = 0$

$\therefore x = 2$

If we had divided by x, we would've lost this solution!

- Evaluate: $x^2 - x - 6$

(a) If $x = -3$ & (b) If $x = -2$

(a) If $x = -3$: $x^2 - x - 6$

$= (-3)^2 - (-3) - 6$

$= 9 + 3 - 6$

$= 6$ ◄

(b) If $x = -2$: $x^2 - x - 6$

$= (-2)^2 - (-2) - 6$

$= 4 + 2 - 6$

$= 0$ ◄

Here we are finding the value of the expression for various values of x

- Given the equation: $x^2 - x - 6 = 0$

(a) Is -3 a root? (b) Is -2 a root?

(a) If $x = -3$: $x^2 - x - 6 \neq 0$ (*see lhs*)

$\therefore$ No, it is not a root

(b) If $x = -2$: $x^2 - x - 6$ does $= 0$ (*see lhs*)

$\therefore$ Yes, it is a root

Here we are testing the truth of the statement that says:

$x^2 - x - 6$ must $= 0$

A root is a value of x that makes this statement true.

Is there another value of x which would make the expression $x^2 - x - 6$ have a value of 0?

- Solve the equation $x^2 - x - 6 = 0$:

$\therefore (x - 3)(x + 2) = 0$

$\therefore x = 3$ or $x = -2$

So, which is the 'other root'? And, why?

Show that $x = 3$ is a root, i.e. check that $x = 3$ makes the equation true:

When $x = 3$: $x^2 - x - 6$ ◄ **the expression**

$= 3^2 - 3 - 6$

$= 9 - 9$

$= 0$ ◄ **the equation**

$\therefore$ **The statement**: $x^2 - x - 6 \mathbf{= 0}$ is true when $x = 3$

$\therefore$ 3 *is* the 'other root'

Nature of the roots

The **nature** of the roots of a quadratic equation means the **type of number** that the roots are and the **number of roots**.

Observe the nature of the roots in the following 4 cases:

Worked Example

Apply the formula to obtain the roots of the following equations:

NB: Write all equations in standard form first.

The Equation: $\mathbf{ax^2 + bx + c = 0}$ **The Roots:** $x = \dfrac{-b \pm \sqrt{\mathbf{b^2 - 4ac}}}{2a}$

(a) $x^2 + 3 = x$

$\therefore\ x^2 - x + 3 = 0$

$\therefore\ \mathbf{a} = 1\ ;\ \mathbf{b} = -1\ ;\ \mathbf{c} = 3$

$x = \dfrac{-(-1) \pm \sqrt{\mathbf{(-1)^2 - 4(1)(3)}}}{2(1)}$

$= \dfrac{1 \pm \sqrt{\mathbf{-11}}}{2}$ ⬅ -11 is problematic! $\sqrt{-11}$ is imaginary.

The roots are imaginary i.e. There are no real roots.

(b) $-x^2 + 6x - 5 = 0$

$\times(-1)\ \therefore\ x^2 - 6x + 5 = 0$

$\therefore\ \mathbf{a} = 1\ ;\ \mathbf{b} = -6\ ;\ \mathbf{c} = 5$

$x = \dfrac{-(-6) \pm \sqrt{\mathbf{(-6)^2 - 4(1)(5)}}}{2(1)}$

$= \dfrac{6 \pm \sqrt{\mathbf{16}}}{2}$ ⬅ 16 is 'nice'. It is a perfect square!

$= \dfrac{6 \pm 4}{2}$

$= \dfrac{10}{2}$ or $\dfrac{2}{2}$

$= (5)$ or (1)

There are 2 roots that are real and rational.

(c) $4x^2 + 6x - 8 = 0$

$2x^2 + 3x - 4 = 0$

$\therefore\ \mathbf{a} = 2\ ;\ \mathbf{b} = 3\ ;\ \mathbf{c} = -4$

$x = \dfrac{-3 \pm \sqrt{\mathbf{3^2 - 4(2)(-4)}}}{2(2)}$

$= \dfrac{-3 \pm \sqrt{\mathbf{41}}}{4}$ ⬅ 41 is 'ok' because $\sqrt{41}$ exists, but it is irrational.

$\simeq (0{,}85)$ or $(-2{,}35)$

There are 2 roots that are real and irrational.

(d) $4x + \dfrac{1}{x} = 4$

$\times x)\ \therefore\ 4x^2 - 4x + 1 = 0$

$\therefore\ \mathbf{a} = 4\ ;\ \mathbf{b} = -4\ ;\ \mathbf{c} = 1$

$x = \dfrac{-(-4) \pm \sqrt{\mathbf{(-4)^2 - 4(4)(1)}}}{2(4)}$

$= \dfrac{4 \pm \sqrt{\mathbf{0}}}{8}$ ⬅ $\sqrt{0} = 0$ so, the roots are the same!

$= \dfrac{4 \pm 0}{8}$

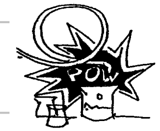
$= \dfrac{1}{2} + 0$ or $\dfrac{1}{2} - 0$

$= \left(\dfrac{1}{2}\right)$ only

There is only 1 root, or, we say, the roots are equal. 'They' are real & rational too.

The discriminant, Δ (Delta)

Delta is Dynamite! . . . Why?



$\mathbf{\Delta = b^2 - 4ac}$ and is the part of the formula which is under the $\sqrt{\ }$ sign.

$\therefore$ The roots of $ax^2 + bx + c = 0$ are $\dfrac{\mathbf{-b + \sqrt{\Delta}}}{\mathbf{2a}}$ and $\dfrac{\mathbf{-b - \sqrt{\Delta}}}{\mathbf{2a}}$

$\mathbf{\Delta}$ impacts crucially on the nature of the roots because of its position:

- under the $\sqrt{\ }$ sign, and
- in the only term where the roots differ

The Possibilities for Δ	The Nature of the Roots
I $\mathbf{\Delta < 0}$, e.g. $\Delta = -16$	imaginary *(no real roots)*
II $\mathbf{\Delta \geq 0}$	real
$\mathbf{\Delta = 0}$	real and equal *(only 1 root)*
$\mathbf{\Delta > 0}$	real and unequal *(2 roots)*
• **$\Delta > 0$ & a perfect square** e.g. $\Delta = 16$	real, rational and unequal *(2 roots)*
• **$\Delta > 0$ & not a perfect square** e.g. $\Delta = 20$	real, irrational and unequal *(2 roots)*

NB: $\mathbf{\Delta = b^2 - 4ac}$ where the values of **a**, **b** and **c** are determined from the standard form of the equation, $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$. So:

Step 1: Write the equation in its standard form.

Step 2: Calculate Δ, the discriminant.

Step 3: Describe the nature of the roots.

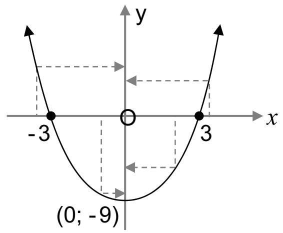
Quadratic Inequalities

The Parabola

The parabola gives a picture of all the possible values of a quadratic expression.

e.g. **$y = x^2 - 9$**

Note: $x^2 - 9$ is y



Recall:

The y-intercept: (0; -9)
(Put x = 0)

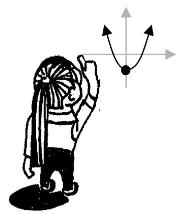
The x-intercepts: $x^2 - 9 = 0$
(Put y = 0) $\therefore\ (x + 3)(x - 3) = 0$
$\therefore\ x = -3$ or 3

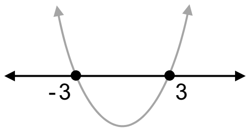
Note:
- The x-intercepts (or roots) of the parabola are the roots of the equation $x^2 - 9$ **= 0**. These root values are the values used when determining the SIGN of $x^2 - 9$ (see below);
- The values of $x^2 - 9$ are read on the y-axis.

The SIGN of $x^2 - 9$: zero, positive or negative, for various values of x

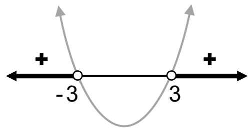
Read, off the graph, for values of x from left to right, the sign of $x^2 - 9$:

- If $x = -3$ or 3: $x^2 - 9$ equals zero . . . *on the x-axis*
- If $x < -3$: $x^2 - 9$ is positive . . . *above the x-axis*
- If $-3 < x < 3$: $x^2 - 9$ is negative . . . *below the x-axis*
- If $x > 3$: $x^2 - 9$ is positive . . . *above the x-axis*

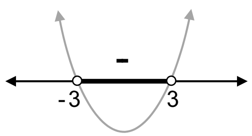




y = 0 for $x = -3$ or 3



y > 0 for $x < -3$ or $x > 3$



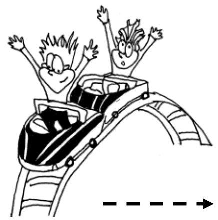
y < 0 for $-3 < x < 3$

Use parabolas to solve the following quadratic equations and inequalities:

$x^2 - 16 = 0$	$x^2 - 12 = 0$	$x^2 = 0$	$x^2 + 2 = 0$
$x^2 - 16 > 0$	$x^2 - 12 > 0$	$x^2 > 0$	$x^2 + 2 > 0$
$x^2 - 16 < 0$	$x^2 - 12 < 0$	$x^2 < 0$	$x^2 + 2 < 0$

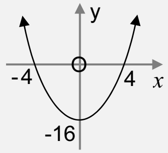
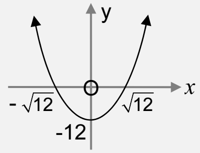
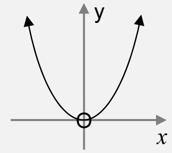
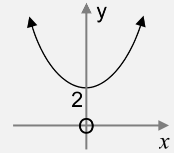
The equations of the parabolas:

$y = x^2 - 16$ | $y = x^2 - 12$ | $y = x^2$ | $y = x^2 + 2$ – – – – →



Given the sign of the expression, determine the values of x.

Consider the following expressions and use sketches to determine the values of x for which the expressions are positive, zero or negative.

y:	$x^2 - 16$	$x^2 - 12$	x^2	$x^2 + 2$
The sketch: **The sign of y:** ↓	 $y = x^2 - 16$	 $y = x^2 - 12$	 $y = x^2$	 $y = x^2 + 2$
y = 0 for:	$x = \pm 4$	$x = \pm\sqrt{12}$	$x = 0$	no values of x
y > 0 for:	$x < -4$ or $x > 4$	$x < -\sqrt{12}$ or $x > \sqrt{12}$	all x except $x = 0$	all x
y < 0 for:	$-4 < x < 4$	$-\sqrt{12} < x < \sqrt{12}$	no values of x	no values of x

Worked Example 1

Solve the following equations and inequalities for x:

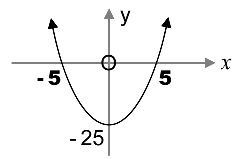
1. $x^2 - 25 = 0$ Hint: Sketch the graph $y = x^2 - 25$

2. (a) $x^2 - 25 < 0$ (b) $x^2 - 25 > 0$ (c) $x^2 - 25 \leq 0$ (d) $x^2 - 25 \geq 0$

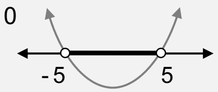
Answers

1. $(x + 5)(x - 5) = 0$
$\therefore$ **$x = -5$ or 5** ◄

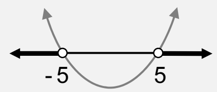
The sketch: **$y = x^2 - 25$:**



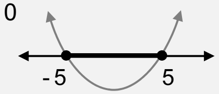
2. (a) $(x + 5)(x - 5) < 0$
$-5 < x < 5$ ◄



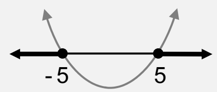
(b) $(x + 5)(x - 5) > 0$
$x < -5$ or $x > 5$ ◄



(c) $(x + 5)(x - 5) \leq 0$
$-5 \leq x \leq 5$ ◄



(d) $(x + 5)(x - 5) \geq 0$
$x \leq -5$ or $x \geq 5$ ◄



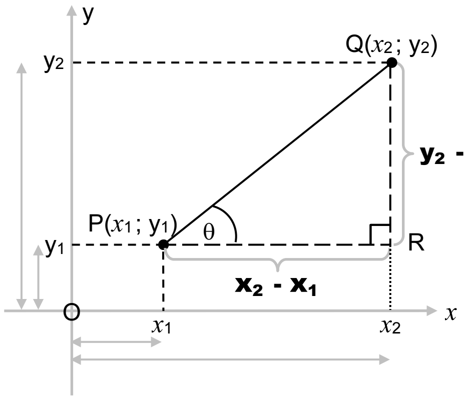
Checklist: The Drawers of Tools

Consider 4 'drawers' of tools - all BASIC FACTS. Use these to analyse the sketches, to reason, calculate, prove . . . .

Distance, Midpoint & Gradient

NB: Bear in mind **Case 1**, **Case 2**, and **Case 3** on page 5.6 & 5.7

For any two fixed points, P(x_1; y_1) & Q(x_2; y_2)



Note:

Vertical length QR = $\mathbf{y_2 - y_1}$

Horizontal length PR = $\mathbf{x_2 - x_1}$

θ is the angle of inclination of the line PQ

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$$

1 Distance PQ . . .

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \Rightarrow PQ = \sqrt{(\quad)^2 + (\quad)^2}$$

the sum of the squares! (Pythag.)

2 Gradient PQ . . .

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} \ (= \tan\theta)$$

change in y / change in x

3 Midpoint of PQ . . .

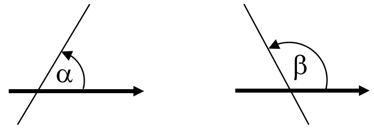
$$\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Average of the x's & of the y's

Parallel lines, Perpendicular lines & Collinearity

- AB || CD ⬌ $m_{AB} = m_{CD}$
- AB ⊥ CD ⬌ $m_{AB} = -\frac{1}{m_{CD}}$ . . . $m_{AB} = -\frac{1}{m_{CD}}$ also means: $m_{AB} \times m_{CD} = -1$
- A, B and C are collinear points ⬌ $m_{AB} = m_{AC}$; $m_{AB} = m_{BC}$; $m_{AC} = m_{BC}$

The Angle of Inclination of a line



The ∠ of inclination of a line is the ∠ which the line makes with the positive direction of the x-axis.

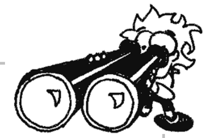
NB: If α or β is the angle of inclination (measured in degrees), then the gradient of the line = tan α or tan β (which is a ratio or number).

∴ Given α or β, one can find **the gradient**: . . . **a number**
Or, given the gradient, one can find α or β: . . . **an angle** (measured in degrees)

Equations of lines

NB: Bear in mind **Case 1**, **Case 2,** and **Case 3** on page 5.9.

- **Standard forms:**
  - General: $\mathbf{y = mx + c}$ or $\mathbf{y - y_1 = m(x - x_1)}$
  - $\mathbf{y = mx}$ . . . when c = 0 . . . lines through the origin
  - $\mathbf{y = c}$ . . . when m = 0 . . . lines || x-axis
  - $\mathbf{x = k}$ . . . lines || y-axis



Finding the equation of a line: Special focus
- through 2 given points . . . find m first
- through 1 point and || or ⊥ to a given line . . . substitute m and the point.

the gradient — $y = \mathbf{m}x + c$ — the point

we need
- **the gradient** &
- **a point**

the gradient — $y - y_1 = \mathbf{m}(x - x_1)$ — the point

- **Y-cuts and X-cuts:** Put $x = 0$ and $y = 0$, respectively.
- **Point of intersection of 2 graphs:**
  Solve the equations of the graphs simultaneously.

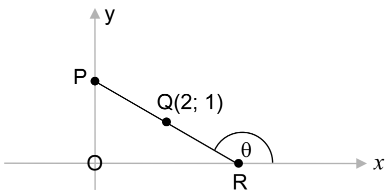
- **If a point lies on a line, the equation is true for it,**
  and, vice versa . . .
  If a point satisfies the equation of a line, the point lies on the line.
  e.g. If a line has the equation $y = x + 1$, then all points on the line can be represented by (x; $x + 1$).

5 ANALYTICAL GEOMETRY

EXERCISE 5.3: Mixed Exercise

Answers on page A5.2

1. Q(2; 1) is the midpoint of line segment PR. P is a point on the y-axis and R is a point on the x-axis.



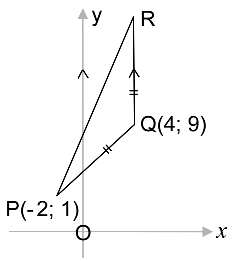
Calculate:

1.1 the coordinates of P and R
1.2 the length of PR (in simplified surd form)
1.3 the gradient of PR
1.4 θ, the ∠ of inclination of PR
1.5 the equation of PR

2. △PQR is isosceles, with PQ = QR and QR || the y-axis.

Calculate:

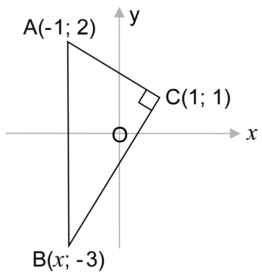
2.1 the length of PQ
2.2 the coordinates of R



3. A, B and C are three points in the Cartesian plane. AC ⊥ BC.

Calculate:

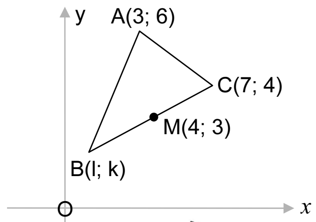
3.1 the gradient of AC
3.2 the value of x
3.3 the length of AB
3.4 the area of △ABC



4. Find the numeral value of k in each of the following cases:

4.1 The straight line $y = 2x + 3$ is parallel to the straight line $2y - kx = 16$.
4.2 Points P(1; -3), Q(4; 6) and R(k; k) are collinear.

5. A(3; 6), B(1; k) and C(7; 4) are the vertices of a triangle with M(4; 3) the midpoint of BC.

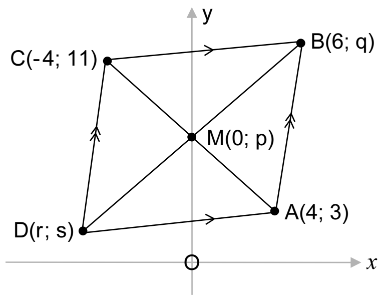


5.1 Determine the value of k.
5.2 If R is the midpoint of AC, prove that

5.2.1 $MR = \frac{1}{2}BA$, and
5.2.2 MR || BA

6. A(4; 3), B(6; q), C(-4; 11) and D(r; s) are the vertices of a parallelogram.

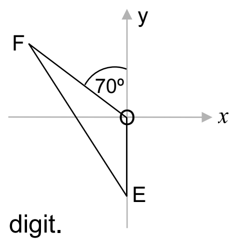
The diagonals intersect at M(0; p). The equation of AB is $y = 5x - 17$.



6.1 Determine the values of p, q, r and s.
6.2 Hence, using the values obtained in Question 6.1, prove whether CA ⊥ DB.
6.3 Conclude which type of quadrilateral ABCD is.

7. △EOF is drawn so that O is at the origin and E is a point on the negative y-axis.

OF makes an angle of 70º with the y-axis.

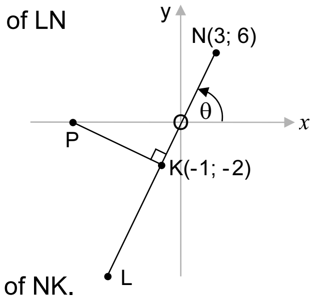


7.1 Find the gradient of OF, rounded off to one decimal digit.
7.2 Find the size of $O\hat{E}F$, rounded off to one decimal digit, if the gradient of EF is $-\frac{3}{2}$.

8. K(-1; -2) is the midpoint of LN with N(3; 6).

PK ⊥ LN with P on the x-axis.

The angle of inclination of NL is θ.

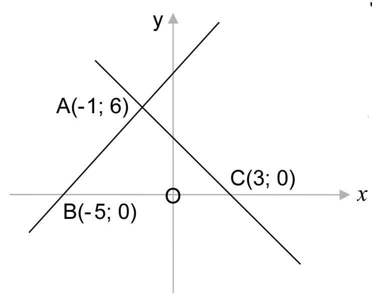


8.1 Determine:

8.1.1 the gradient of NK.
8.1.2 the gradient of PK.
8.1.3 the size of θ, rounded off to one decimal digit.
8.1.4 the coordinates of L.
8.1.5 the equation of PK.
8.1.6 the coordinates of P.

8.2 Determine the equation of the straight line parallel to PK and which passes through the origin.

9. BA and CA intersect at A(-1; 6).



9.1 Is line BA perpendicular to line AC? Show clearly all working details to justify your answer.
9.2 Calculate the size of $A\hat{B}C$ (rounded off to one decimal digit).
9.3 Calculate the length of AB. (Leave your answer in simplified surd form.)
9.4 Show that △ABC is isosceles.
9.5 If A, B and D(a; 8) are three collinear points, calculate the value of a.

▶ 3 General forms of the equation of a parabola

There are two other forms of the equation of a parabola:

- ▶ The 'root form': $\mathbf{y = a(x - A)(x - B)}$, and
- ▶ The 'standard form': $\mathbf{y = ax^2 + bx + c}$

The 3 General forms are:

Turning point form	Root form	Standard form
$\mathbf{y = a(x - p)^2 + q}$	$\mathbf{y = a(x - A)(x - B)}$	$\mathbf{y = ax^2 + bx + c}$
t.p. (p; q)	*roots A & B*	*y-intercept*

We use basic facts and symmetry to determine the axis-intercepts and the turning point in each form.

▶ The turning point form of the equation: $y = a(x - p)^2 + q$

- **The y-intercept**
  Substitute $x = 0$
- **The x-intercept(s)**
  Substitute $y = 0$
- **The turning point**
  Read the turning point off the equation

▶ The 'root form' of the equation: $y = a(x - A)(x - B)$

- **The y-intercept**
  Substitute $x = 0$: $y = a(\mathbf{0} - A)(\mathbf{0} - B)$
- **The x-intercept(s) . . .**
  Substitute $y = 0$ ➡ $a(x - A)(x - B) = \mathbf{0}$ ➡ $\therefore\ x = \mathbf{A}$ or $x = \mathbf{B}$
- **The turning point**
  The **x-coordinate** *(the axis of symmetry value of x):*
  - ▹ The axis of symmetry can be found halfway between the roots, or by finding the average of the roots . . . $\mathbf{x = \frac{A + B}{2}}$

  The **y-coordinate** *(the minimum/maximum y-value):*
  - ▹ The minimum or maximum value of the graph is found by substituting the 'symmetry x-value' in the equation.

▶ The standard form of the equation: $y = ax^2 + bx + c$

- **The y-intercept**
  The y-intercept of a graph occurs when $\mathbf{x = 0}$:
  $\therefore$ The y-intercept: $y = a(\mathbf{0})^2 + b(\mathbf{0}) + c = \mathbf{c}$
  $\therefore$ The point where the graph cuts the y-axis is $(\mathbf{0};\ \mathbf{c})$

- **The x-intercept(s)**
  The **x-intercept(s)**, if any, of a graph occur(s) when $\mathbf{y = 0}$:
  So, $\mathbf{y = 0}$ ➡ $ax^2 + bx + c = \mathbf{0}$ . . . then solve for x
- **The turning point**
  The **x-coordinate** *(the axis of symmetry value of x):*
  - ▹ The axis of symmetry can be found from: the formula, the roots or by completing the square

  The **y-coordinate** *(the minimum/maximum y-value):*
  - ▹ The minimum or maximum value of the graph is found by substituting the 'symmetry x-value' in the equation or by completing the square.

The axis of symmetry

A parabola is always symmetrical about the axis of symmetry.

$\therefore$ The axis of symmetry must be halfway between the roots, i.e. the average of the roots.

Axis of symmetry: $\mathbf{x = \frac{A + B}{2}}$ . . . *where* **A** *&* **B** *are the roots*

The roots of the equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . . .

We can use **Δ** instead of $\mathbf{b^2 - 4ac}$

$\therefore$ The x-intercepts of the graphs are $\left(\frac{-b + \sqrt{\Delta}}{2a};\ \mathbf{0}\right)$ & $\left(\frac{-b - \sqrt{\Delta}}{2a};\ \mathbf{0}\right)$

$\therefore$ The sum of the roots:

$$\mathbf{A + B} = \frac{-b + \sqrt{\Delta}}{2a} + \frac{-b - \sqrt{\Delta}}{2a} = \frac{-b + \sqrt{\Delta} - b - \sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$\therefore$ The average of the roots:

$$\mathbf{\frac{A + B}{2} = -\frac{b}{2a}}$$

. . . **Axis of symmetry: $\mathbf{x = -\frac{b}{2a}}$**

TRIGONOMETRIC GRAPHS

Point-by-point Plotting

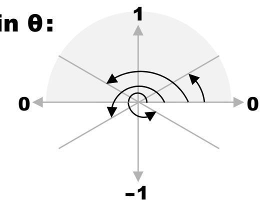
Essential for point-by-point plotting of the basic graphs for $\theta \in [0°; 360°]$:

- Know the critical values of the ratios
- Know the signs of the ratios in the 4 quadrants

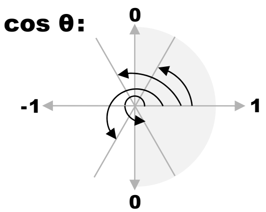
Without the use of a calculator.

'The wheels' combine the critical values and the signs of the ratios.

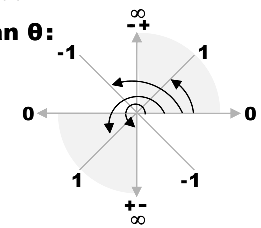
sin θ:



cos θ:



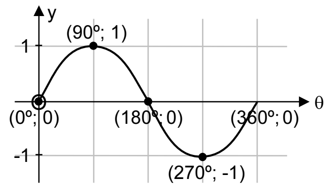
tan θ:



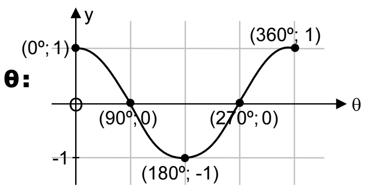
Use the wheels to plot the 'critical points' before drawing the waves.

- Decide where to start: Start at the beginning of the wave.
- Choose the scale for each graph: All the critical values are on the wheels.

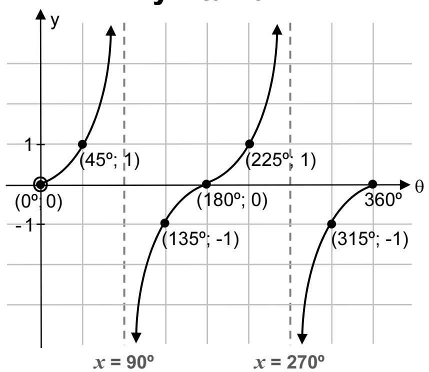
y = sin θ:



y = cos θ:

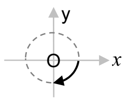


y = tan θ:

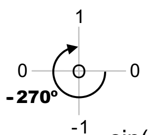


- Extend the domain to: $\theta \in [-360°; 360°]$

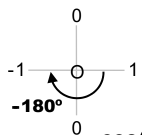
Use the wheels to write down the critical values of negative angles by going clockwise:



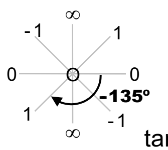
e.g.



sin(-270°) = 1



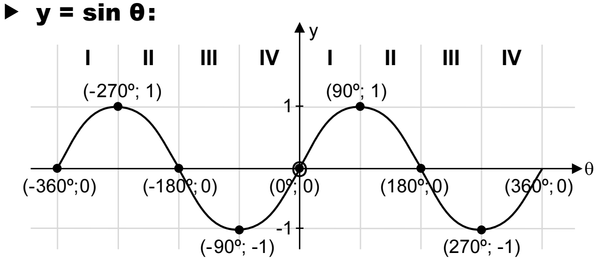
cos(-180°) = -1



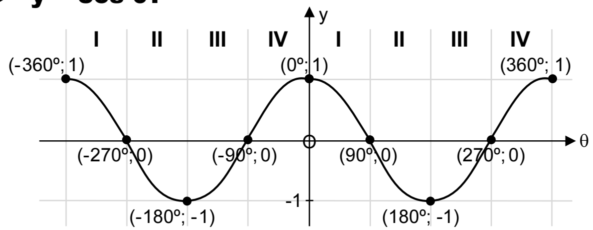
tan(-135°) = 1

Plot points for all 3 basic graphs

- **y = sin θ:**



- **y = cos θ:**



Note: The coordinates of the **turning points**
& The coordinates of the **axis-intercepts**.

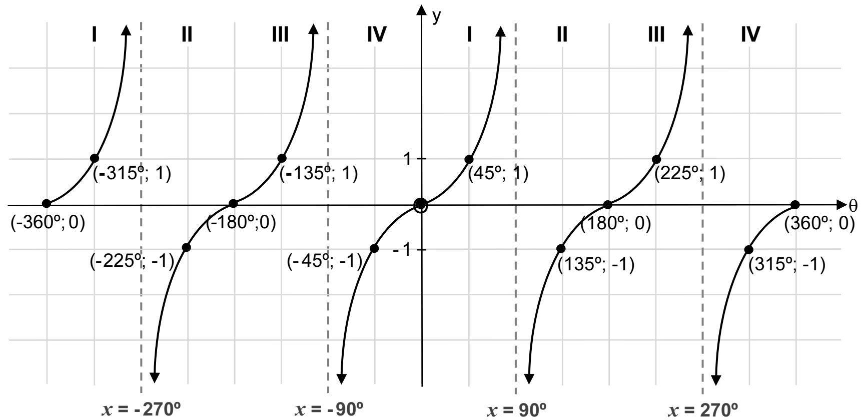
Features of the graphs:

For both y = sin θ & y = cos θ:

- **The amplitude** = 1 unit
- **The range**: $-1 \leq y \leq 1$
- **The maximum value** = 1
- **The minimum value** = -1
- **The period** = 360°

Consider each wave as being divided into **4 quarters**

- **y = tan θ:**



- **The range:** $y \in \mathbb{R}$
- **The period** = 180°
- **The asymptotes:** $x = -270°$; $x = -90°$; $x = 90°$; $x = 270°$

FUNCTIONS & GRAPHS 6

Trigonometry Unlimited: All Angles!

Considering all possible angles, including negative angles

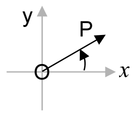
In considering the domain [0º; 360º], we have considered all 4 quadrants, but only angles rotating anticlockwise, and only 1 revolution.
Rotating anticlockwise or clockwise, through any number of revolutions covers the same 4 quadrants with the same results!

General forms: (θ ± 360º) ; (-θ)

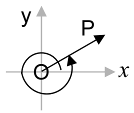
▶ Rotating through revolutions, anticlockwise and clockwise

e.g. (1) Compare: **30º** ; **390º** and **-330º**

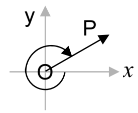
▶ **30º**



▶ 390º = **30º + 360º**



▶ -330º = **30º - 360º**



These are 3 different ∠s, but they are 'co-terminal', i.e. their end arms coincide.

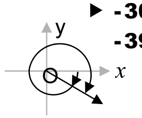
∴ They have the same trig ratios:

$$\sin 30^\circ = \sin 390^\circ = \sin(-330^\circ) = \frac{1}{2}$$

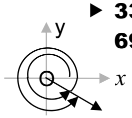
(2) Compare: **330º** ; **690º** ; **-30º** ; **-390º**

Standard positions:

▶ **-30º** ; **-390º**



▶ **330º** ; **690º**



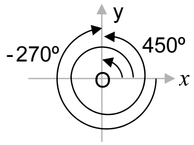
These angles are co-terminal.
∴ They have the same trig ratios.

$$\sin 330^\circ = \sin 690^\circ = \sin(-30^\circ) = \sin(-390^\circ) = -\frac{1}{2}$$

(3) Compare: **90º** ; **450º** ; **-270º**

450º = 90º **+ 360º**

-270º = 90º **- 360º**



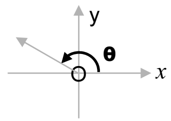
These 3 angles are co-terminal, i.e. they have the same terminal arm, OY.
The sine of all 3 angles is 1, their cosine is 0 and their tan is undefined.

We conclude the following reduction formulae for θ ± 360º:

sin(θ + 360º) = sin θ	cos(θ + 360º) = cos θ	tan(θ + 360º) = tan θ
sin(θ - 360º) = sin θ	cos(θ - 360º) = cos θ	tan(θ - 360º) = tan θ

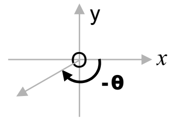
▶ Rotating clockwise

If **θ** is



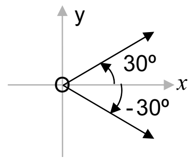
anticlockwise rotation

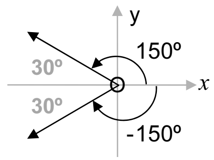
then **-θ** is

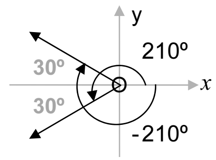


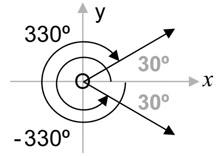
clockwise rotation

▸ Compare **30º** ; **150º** ; **210º** ; **330º**
to: **-30º** ; **-150º** ; **-210º** ; **-330º**







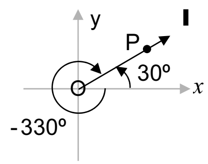


The reduction formulae for -θ: *Consider θ acute*

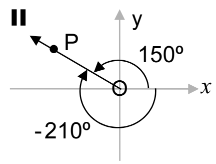
sin(-θ) = **-**sin θ	cos(-θ) = **+**cos θ	tan(-θ) = **-**tan θ
Compare to: sin(360º - θ) = **-**sin θ	cos(360º - θ) = **+**cos θ	tan(360º - θ) = **-**tan θ

▸ The following pairs of angles are **co-terminal**.
∴ They have the same trigonometric ratios.

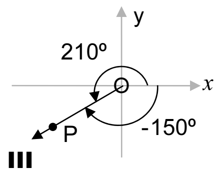
30º & -330º



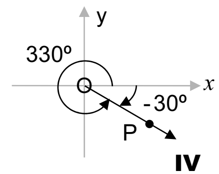
150º & -210º



210º & -150º



330º & -30º



I: sin(-330º) = sin 30º = $\frac{1}{2}$

II: sin(-210º) = sin 150º = $\frac{1}{2}$

III: sin(-150º) = sin 210º = $-\frac{1}{2}$

IV: sin(-30º) = sin 330º = $-\frac{1}{2}$

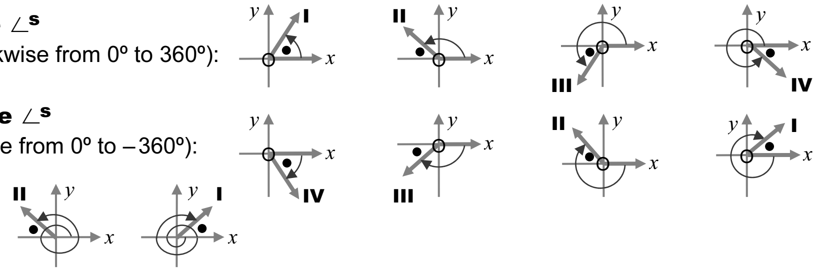
So: sin (-330º) = + sin 30º
sin (-210º) = + sin 30º
sin (-150º) = - sin 30º
sin (-30º) = - sin 30º

TRIG SUMMARY (Grade 11)

▸ ANGLES IN STANDARD POSITIONS

- **Positive ∠s** (anticlockwise from 0º to 360º):
- **Negative ∠s** (clockwise from 0º to –360º):

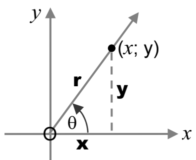
Also possible:



▸ THE RATIOS

& their

- **Definitions:**

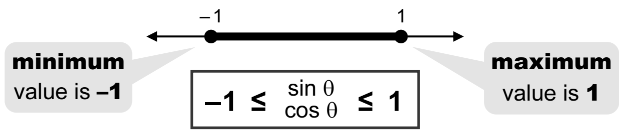


- **Signs:**
  sin θ is positive in **I** & **II**
  cos θ is positive in **I** & **IV**
  tan θ is positive in **I** & **III**

- **Critical values:**

	sin θ	cos θ	tan θ
Definitions	$\frac{y}{r}$ ($\frac{O}{H}$)	$\frac{x}{r}$ ($\frac{A}{H}$)	$\frac{y}{x}$ ($\frac{O}{A}$)
Signs	II, I	I, IV	I, III
Critical values	1; 0, 0; –1	0; –1, 1; 0	–1, ∞, 1; 0, 0; 1, ∞, –1 (– +, + –)

- **Minimum & Maximum values** of **sin θ** & **cos θ**:
  The values of **sin θ** & **cos θ** range from **–1** to **1**.



$$-1 \leq \begin{matrix}\sin\theta \\ \cos\theta\end{matrix} \leq 1$$

All values are **proper fractions** or **0** or **±1**.

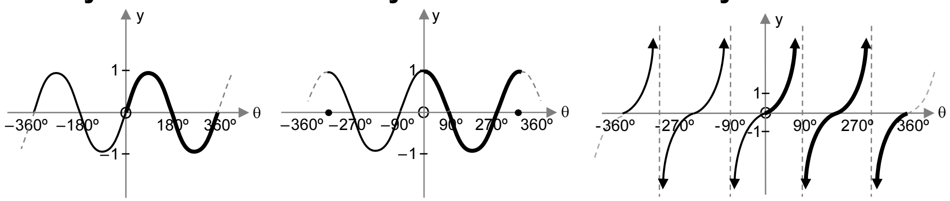
tan θ

Note that tan θ has **no minimum or maximum** values.

The **range of values** of tan θ is from –∞ to ∞.

- **Graphs:**

y = sin θ **y = cos θ** **y = tan θ**



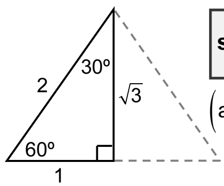
▸ IDENTITIES

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

∴ $\sin^2\theta = 1 - \cos^2\theta$
& $\cos^2\theta = 1 - \sin^2\theta$

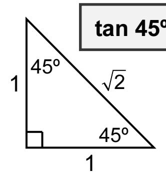
▸ SPECIAL ∠s

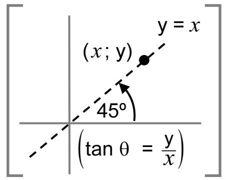


$\sin 30º = \frac{1}{2}$

(and $\cos 60º = \frac{1}{2}$)

$\tan 45º = 1$





& THEIR 'FAMILIES':

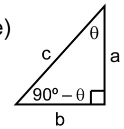
	30º 'family'	60º 'family'	45º 'family'	
II	150º	120º	135º	◂ 180º – ●
III	210º	240º	225º	◂ 180º + ●
IV	330º	300º	315º	◂ 360º – ●

▸ GENERAL FORMS

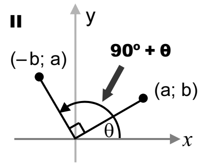
ANY ratio $\left(\begin{matrix}180º \pm \theta \\ 360º - \theta \\ -\theta\end{matrix}\right)$ = **±** that **SAME** ratio of **θ**

▸ CO-RATIOS (sine and **co**sine)

- **90º – θ** (an **acute** angle)
  sin(90º – θ) = cos θ
  cos(90º – θ) = sin θ



- **90º + θ** (an **obtuse** angle)
  sin(90º + θ) = cos θ
  cos(90º + θ) = –sin θ



The ratio **CHANGES** to the **CO**-ratio.

▸ SOLUTION OF Δs

In Right-angled Δs, we use:

- Regular trig. ratios
- the Theorem of Pythagoras
- Area = $\frac{1}{2}$bh

In Non-Right-angled Δs, we use:

- **Sine Rule:** $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- **Cosine Rule:** $c^2 = a^2 + b^2 - 2ab\cos C$
- **Area Rule:** **AREA** = $\frac{1}{2}$ab sin C

But also: Area of Δ = $\frac{1}{2}$bh

A Vital Concept: The word, subtend . . .

Understanding the word **'subtend'** is crucial to understanding circle geometry.

We say that an arc or a chord **subtends** angles at the centre or at the circumference of a circle (although it could do so at other points).

Central and Inscribed angles

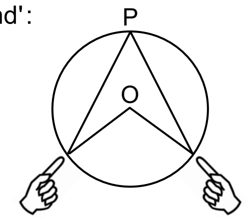
Study figures 1 to 4 at the bottom of the page.

In all the figures, arc AB ($\overset{\frown}{AB}$), or chord AB, **subtends**:

- a **central AÔB** at the **centre** of the circle, and
- an **inscribed AP̂B** at the **circumference** of the circle.

Consider that *subtend* means *support*.

To ensure that you grasp the meaning of the word 'subtend':



- Take **each** of the figures:
  - Place your index fingers on A & B;
  - move along the radii to meet at O and back; then,
  - move to meet at P on the circumference and back.
- Turn your book upside down and sideways. You need to recognise different views of these situations.
- Take note of whether the angles are acute, obtuse, right, straight or reflex.
- Redraw figures 1 to 4 leaving out the chord AB completely and **observe the arc** subtending the central and inscribed angles in each case.

These figures depict the progression of:

- **a growing arc AB** (from minor to major), and
- **the angles** (from acute to reflex) which it subtends at the centre and at the circumference of the circle.

Investigation 2: ∠ at centre, ∠ at circumference

Draw some of your own larger sketches of figures 1 to 4

- What is the range of possible values of AÔB? And of AP̂B?
- Measure **AÔB** and **AP̂B** in each case:

	fig. 1	fig. 2	fig. 3	fig. 4
AÔB =				
AP̂B =				

Is there a relationship between the sizes of AÔB and AP̂B in each figure?

What can you conclude? In words?

Check your conclusions against statement **2.1** *on page 9.7*

- Is there something special about **AP̂B** in **figure 3**?
  Is there something special about **chord AB** in **figure 3**?

What can you conclude? In words?

Check your conclusions against statement **2.2** *on page 9.7*

Observe the **chord AB** vs the **arc AB** on figures 1 to 4

- In figure 1 and 2, **chord AB** subtends AÔB at the centre, but it no longer does so for **AÔB** ≥ 180º *(see figure 3 and figure 4)*. The chord disappears!
- The **arc AB**, however, subtends AÔB and AP̂B no matter their sizes.
  So, when it comes to subtending, we can rely on arcs more readily than chords.

A Progression of Figures 1 - 4 illustrates, as **arc AB** grows:

- *CENTRAL ANGLES*, angles being **subtended** at the **centre**, and
- *INSCRIBED ANGLES*, angles being **subtended** at the **circumference** of a circle.

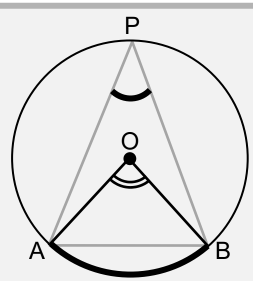


Figure 1

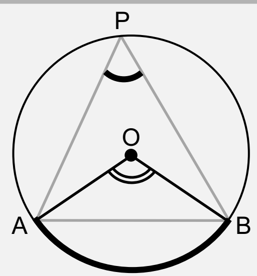


Figure 2

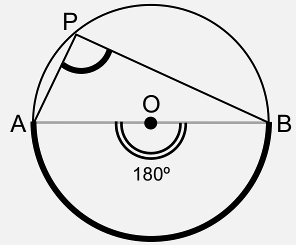


Figure 3

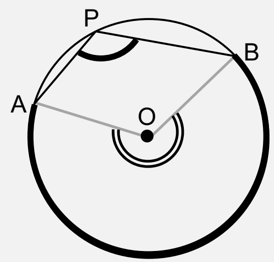


Figure 4

EXERCISE 9.6: Mixed Exercise

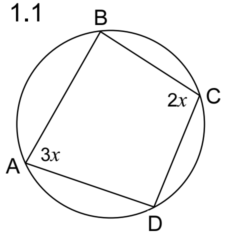
Answers on page A9.9

Answers on page A9.9

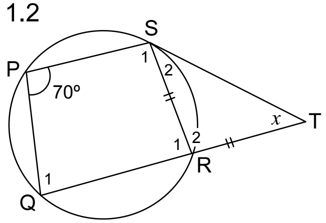
Q1 to Q8: Without tangents

1. Determine, with reasons, the value of x:

1.1

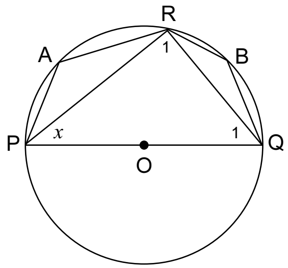


1.2



2.1 O is the centre of the circle. x = 40°

Determine, with reasons, the size of $\hat{Q}_1$, $\hat{B}$ and $\hat{A}$.

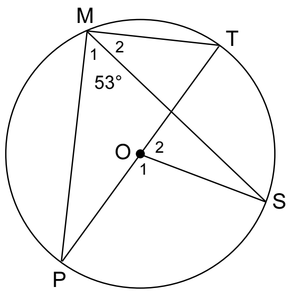


2.2 M, P, S and T are points on a circle with centre O.

PT is a diameter.

MP, MT, MS and OS are drawn. $\hat{M}_1$ = 53°.

Determine, with reasons, the size $\hat{O}_2$.

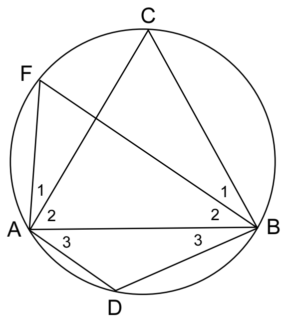


2.3 AB = BC and $A\hat{B}C$ = 50°

Calculate, with reasons, the sizes of

2.3.1 $\hat{F}$ and

2.3.2 $\hat{D}$.



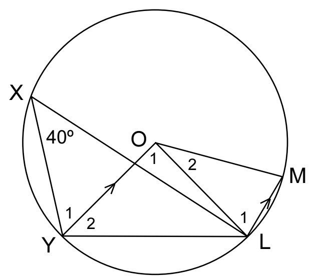
3. OY || ML and $\hat{X}$ = 40°.

Calculate, with reasons, the sizes of the following:

3.1 $\hat{O}_1$

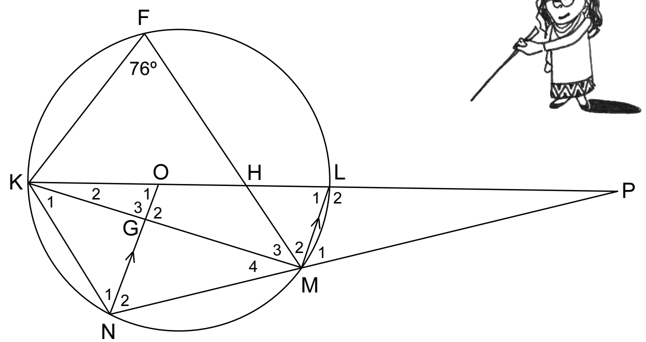
3.2 $\hat{Y}_2$

3.3 $\hat{O}_2$



4. O is the centre of the circle and diameter KL is produced to meet NM produced at P.

ON || LM and $\hat{F}$ = 76°.



Calculate, giving reasons, the sizes of:

4.1 $\hat{L}_1$

4.2 $\hat{O}_1$

4.3 $\hat{M}_4$

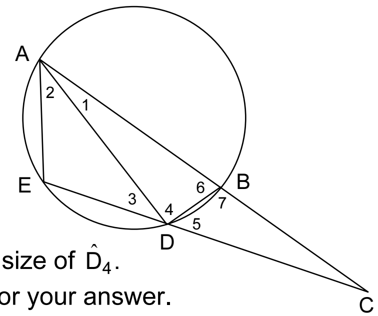
4.4 $\hat{N}_1 + \hat{N}_2$

4.5 $\hat{M}_1$

4.6 Prove that KG = GM

5. AB is a diameter of the circle.

The chord ED and the diameter AB are produced to meet at C.



5.1 Write down the size of $\hat{D}_4$.
Give a reason for your answer.

5.2 If $\hat{A}_1$ = 22° and $\hat{C}$ = 24°, calculate the size of $\hat{D}_5$ and deduce that DA bisects $B\hat{A}E$.

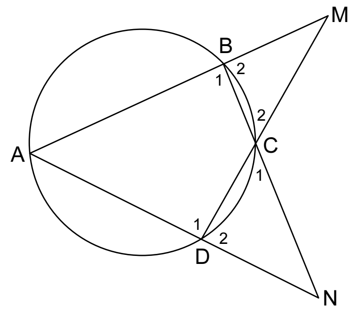
6.1 Complete the following by writing the appropriate missing word.

If a chord of a circle subtends a right angle on the circumference, then this chord is a . . . . . .

6.2 A, B, C and D are points on the circle.

BC produced and AD produced meet at N.

AB produced and DC produced meet at M.



If $\hat{M} = \hat{N}$, prove that AC is a diameter of the circle.

Hint: Let $\hat{M} = x$ and $\hat{C}_1$ = y

THEOREMS AND FACTS

ACT!

A: Be **A**ctive

C: Use all your **C**lues

T: Apply the **T**heory systematically, recalling each group, and fact, one at a time.

There are 4 groups of theorems and facts

- the **centre** group
- the **'no centre'** group
- the **cyclic quadrilateral** group, &
- the **tangent** group

10 TRIGONOMETRY (Part 2) – area, sine and cosine rules

TOPIC OUTLINE

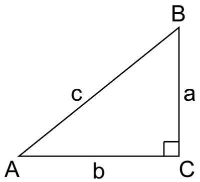
In this module we will examine methods to calculate:

- ❐ the **area**, and
- ❐ the **sides** and the **angles** of all triangles:

❐ THE AREA OF A TRIANGLE

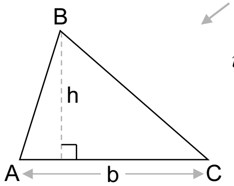
▶ **A = $\frac{\text{base} \times \text{height}}{2}$** . . . *This formula can be used in any triangle*

right ∠^{d} Δ:



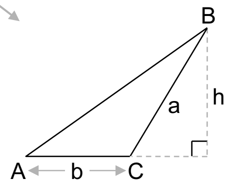
Area = $\frac{ba}{2}$

acute ∠^{d} Δ:



Any 1 of the 3 sides could be used as base. The height must then correspond.

obtuse ∠^{d} Δ:



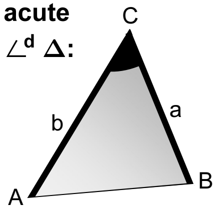
Area = $\frac{bh}{2}$ = $\frac{\text{base} \times \text{height}}{2}$

We will also derive and use a new formula:

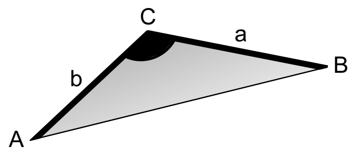
The Area rule:

▶ **A = $\frac{1}{2}$ab sin C**

acute ∠^{d} Δ:



obtuse ∠^{d} Δ:



In words: The area = $\frac{1}{2}$ the product of 2 sides × the sine of the included angle

❐ THE SIDES AND ANGLES OF A TRIANGLE

▶ RIGHT ANGLED Δ^s

Revision of Grade 10

We use:

the **regular trigonometric ratios**, sin θ, cos θ and tan θ, and the **Theorem of Pythagoras**

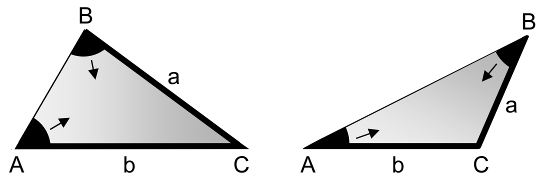
▶ ACUTE- AND OBTUSE-ANGLED Δ^s

We will derive & use:

the new **sine** & **cosine** formulae in the Gr 11 curriculum

The Sine rule:

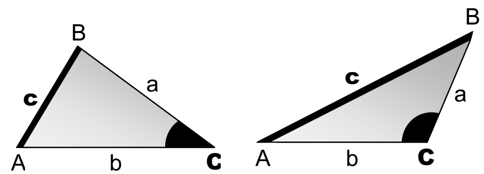
▶ $\frac{\sin A}{a} = \frac{\sin B}{b}$



In words: The sine of an angle over the side opposite it equals the sine of any other angle over the side opposite it.

The Cosine rule:

▶ $c^2 = a^2 + b^2 - 2ab \cos C$



In words: The square of any side (of an acute or obtuse ∠^{d} Δ) equals the sum of the squares of the other 2 sides minus twice the product of these 2 sides and the cos of the included angle.

Independent Events

Two events A and B are said to be **independent** if the outcome of the one does not influence the outcome of the other. Some examples of independent events are: flipping 2 or more coins, throwing 2 or more dice and flipping a coin and throwing a dice. The outcome of flipping one coin does not affect the outcome of flipping another coin. So, too, the outcome of throwing one dice does not affect the outcome of throwing another. **Tree diagrams** are extremely useful for illustrating both independent and not independent events.

Let H be the event of getting a head.

We will consider:

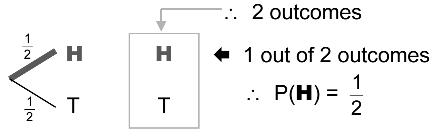
❶ P(H) when flipping 1 coin

❷ P(HH) when flipping 2 coins

❸ P(HHH) when flipping 3 coins

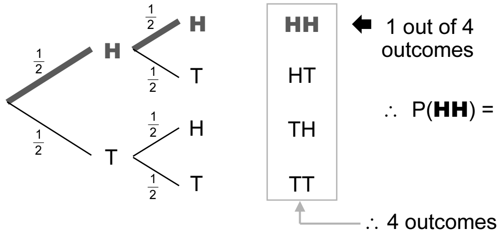
P(H)?

❶ Flipping 1 coin



❷ Flipping 2 coins

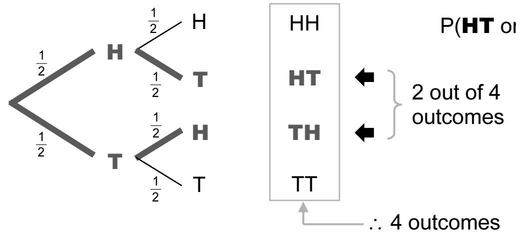
P(H and H)?



∴ The probability of getting a head **and** a head:

P(**H and H**) = $\frac{1}{2}$ **x** $\frac{1}{2}$ = $\frac{1}{4}$ . . . **'AND'** means **MULTIPLY**

P(HT or TH)?



∴ The probability of getting one head and one tail in any order:

P(HT **or** TH) = $\frac{1}{2}$ **+** $\frac{1}{2}$ = $\frac{1}{4}$ . . . **'OR'** means **ADD**

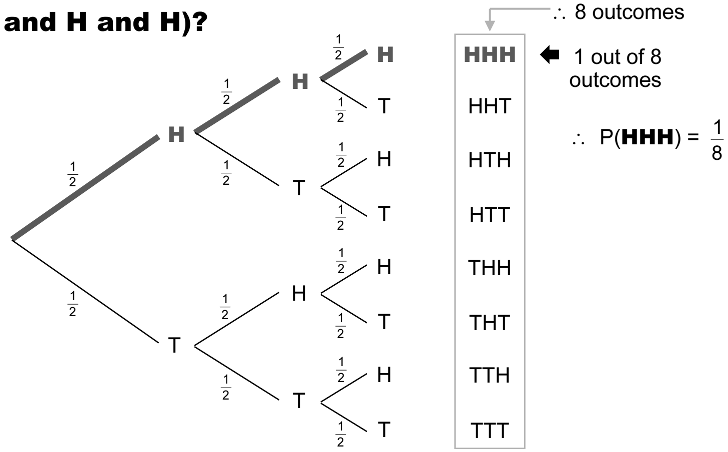
We will consider multiple outcomes:

❷ HT or TH

❸ HTT or THT or TTH

❸ Flipping 3 coins

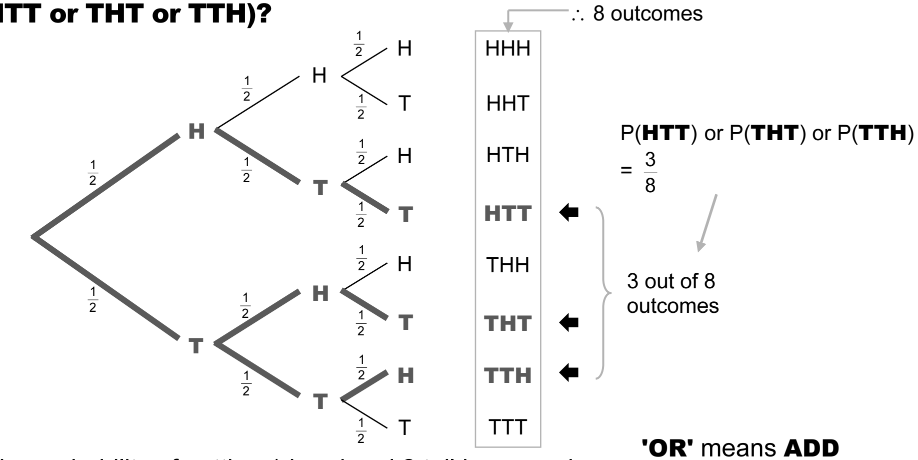
P(H and H and H)?



∴ The probability of getting a head **and** a head **and** a head:

P(**H and H and H**) = $\frac{1}{2}$ **x** $\frac{1}{2}$ **x** $\frac{1}{2}$ = $\frac{1}{8}$ . . . **'AND'** means **MULTIPLY**

P(HTT or THT or TTH)?



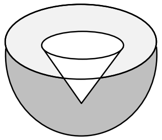
∴ The probability of getting 1 head and 2 tail in any order:

'OR' means **ADD**

P(HTT **or** THT **or** TTH) = $\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$ **+** $\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$ **+** $\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$ = $\frac{3}{8}$

▶ MEASUREMENT [6]

QUESTION 8



A solid metallic hemisphere has a radius of 3 cm. It is made of metal A. To reduce its weight a conical hole is drilled into the hemisphere (as shown in the diagram) and it is completely filled with a lighter metal B. The conical hole has a radius of 1,5 cm and a depth of $\frac{8}{9}$ cm.

Calculate the ratio of the volume of metal A to the volume of metal B. [6]

▶ EUCLIDEAN GEOMETRY [40]

QUESTION 9

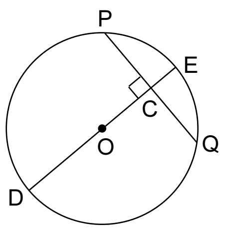
9.1 Complete the statement so that it is valid:

The line drawn from the centre of the circle perpendicular to the chord . . . (1)

9.2 In the diagram, O is the centre of the circle.

The diameter DE is perpendicular to the chord PQ at C.

DE = 20 cm and CE = 2 cm.

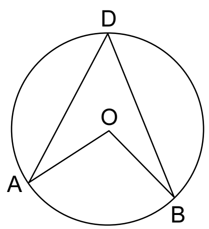


Calculate the length of the following with reasons:

9.2.1 OC 9.2.2 PQ (2)(4) [7]

QUESTION 10

10.1 In the diagram, O is the centre of the circle and A, B and D are points on the circle.

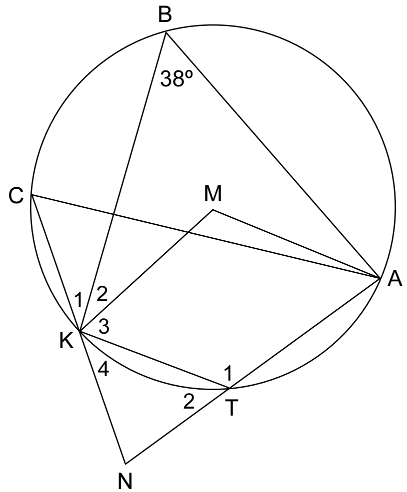


Use Euclidean geometry methods to prove the theorem which states that $A\hat{O}B = 2A\hat{D}B$. (5)

10.2 In the diagram, M is the centre of the circle. A, B, C, K and T lie on the circle.

AT produced and CK produced meet in N.

Also NA = NC and $\hat{B} = 38^\circ$.



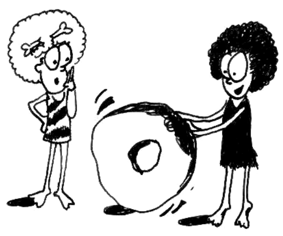
10.2.1 Calculate, with reasons, the size of the following angles:

(a) $K\hat{M}A$ (b) $\hat{T}_2$ (2)(2)

(c) $\hat{C}$ (d) $\hat{K}_4$ (2)(2)

10.2.2 Show that NK = NT. (2)

10.2.3 Prove that AMKN is a cyclic quadrilateral. (3) [18]



QUESTION 11

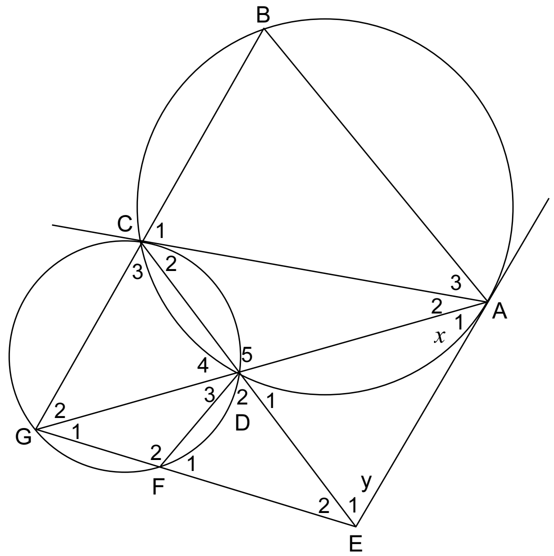
11.1 Complete the following statement so that it is valid:

The angle between a chord and a tangent at the point of contact is . . . (1)

11.2 In the diagram, EA is a tangent to circle ABCD at A.

AC is a tangent to circle CDFG at C.

CE and AG intersect at D.



If $\hat{A}_1 = x$ and $\hat{E}_1 = y$, prove the following with reasons:

11.2.1 BCG || AE (5)

11.2.2 AE is a tangent to circle FED (5)

11.2.3 AB = AC (4) [15]

TOTAL: 150

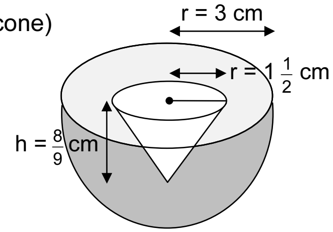
▶ MEASUREMENT [6]

8. Volume of metal B (the cone)

$= \frac{1}{3}\pi r^2 . h$

$= \frac{1}{3}\pi (1,5)^2 . \frac{8}{9}$

$= \frac{2}{3}\pi$



Volume of the hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{2\pi}{3} . 3^3$

$= 18\pi$

$\therefore$ Volume of metal A $= 18\pi - \frac{2}{3}\pi = 17\frac{1}{3}\pi$

$\therefore$ The ratio:

Volume of metal A : Volume of metal B

$= 17\frac{1}{3}\pi : \frac{2}{3}\pi$

$\times 3)$ $= 52\pi : 2\pi$

$\div 2\pi)$ $=$ **26 : 1** ◄

▶ EUCLIDEAN GEOMETRY [40]

9.1 . . . **bisects the chord** ◄

9.2.1 OE = OD $= \frac{1}{2}(20)$ = 10 cm   *radii $= \frac{1}{2}$ diameter*

$\therefore$ **OC = 8 cm** ◄ . . . *CE = 2 cm*

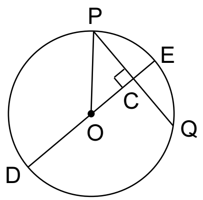
9.2.2 In ΔOPC:

$PC^2 = OP^2 - OC^2$ . . . *Pythagoras*

$= 10^2 - 8^2$

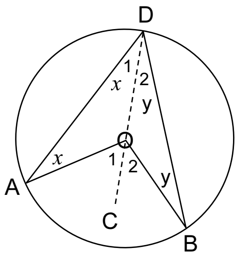
$= 36$

$\therefore$ PC = 6 cm



$\therefore$ **PQ = 12 cm** ◄ . . . *line from centre ⊥ to chord*

10.1 Construction: Join DO and produce it to C



Proof:

Let $\hat{D}_1 = x$

then $\hat{A} = x$ . . . *∠ˢ opp = radii*

$\therefore \hat{O}_1 = 2x$ . . . *ext. ∠ of ΔDAO*

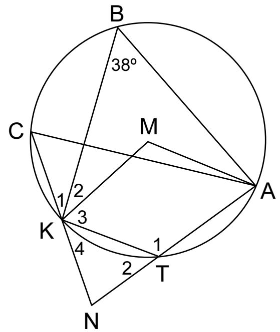
Similarly, $\hat{O}_2 = 2y$

$\therefore A\hat{O}B = 2x + 2y$

$= 2(x + y)$

$=$ **2 A$\hat{D}$B** ◄

10.2



10.2.1 (a) K$\hat{M}$A = 2(38°) . . . *∠ at centre = 2 × ∠ at circumference*

= **76°** ◄

(b) $\hat{T}_2$ = **38°** ◄ . . . *ext. ∠ of cyclic quad. BKTA*

(c) $\hat{C}$ = **38°** ◄ . . . *∠ˢ in same segment or, ext. ∠ of c.q. CKTA*

(d) N$\hat{A}$C = 38° . . . *∠ˢ opposite = sides*

$\therefore \hat{K}_4$ = **38°** ◄ . . . *ext. ∠ of c.q. CKTA*

10.2.2 In ΔNKT: $\hat{K}_4 = \hat{T}_2$ . . . *both = 38° in 10.2.1*

$\therefore$ **NK = NT** ◄ . . . *sides opp = ∠ˢ*

10.2.3 K$\hat{M}$A = 2(38°) . . . *see 10.2.1(a)*

& $\hat{N}$ = 180° - 2(38°) . . . *sum of ∠ˢ of ΔNKT (see 10.2.2)*

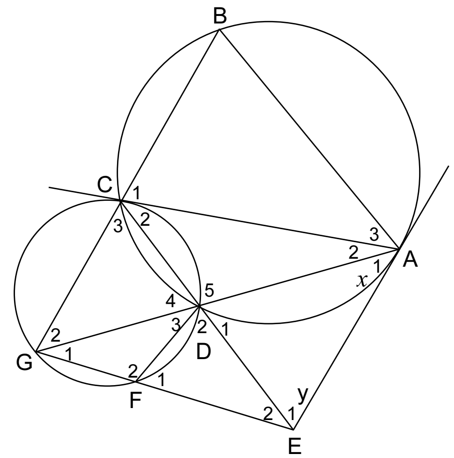
$\therefore$ K$\hat{M}$A + $\hat{N}$ = 180°

$\therefore$ **AMKN is a cyclic quadrilateral** ◄

. . . *opposite ∠ˢ of quad upplementary or conv. opp ∠ˢ of cyclic quad*

11.1 . . . **equal to the angle subtended by the chord in the alternate segment.** ◄

11.2



11.2.1 $\hat{A}_1 = x$ . . . *given*

$\therefore \hat{C}_2 = x$ . . . *tan chord theorem*

$\therefore \hat{G}_2 = x$ . . . *tan chord theorem*

$\therefore \hat{A}_1 = \hat{G}_2$

$\therefore$ **BCG || AE** ◄ . . . *alternate ∠ˢ =*

11.2.2 $\hat{E}_1 = \hat{C}_3 = y$ . . . *alternate ∠ˢ ; BG || AE*

$\therefore \hat{F}_1 = y$ . . . *exterior ∠ of cyclic quad*

$\therefore \hat{E}_1 = \hat{F}_1$

$\therefore$ **AE is a tangent to ⊙FED** ◄

. . . *converse tan-chord theorem*

11.2.3 $\hat{C}_1$ = C$\hat{A}$E . . . *alternate ∠ˢ ; BCG || AE*

$= \hat{B}$ . . . *tan chord theorem*

$\therefore \hat{C}_1 = \hat{B}$

$\therefore$ **AB = AC** ◄ . . . *sides opposite = ∠ˢ*

ANALYTICAL GEOMETRY: TOOLKIT

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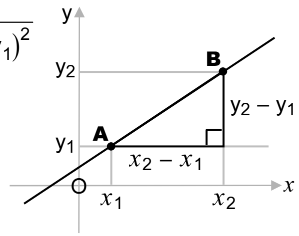
FORMULAE

Consider two points A(x_1; y_1) and B(x_2; y_2):

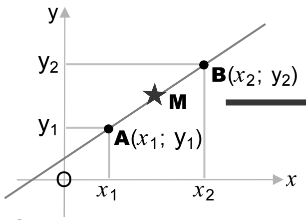
DISTANCE

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ . . . *Thm of Pythagoras*

$\therefore AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



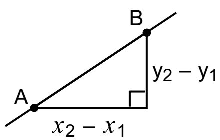
MIDPOINT



$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

The co-ordinates of the midpoint, M, are the **averages** *of the co-ordinates of the endpoints, A and B.*

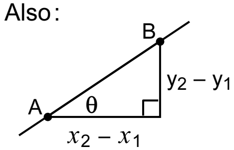
GRADIENT



$m = \frac{\text{change in y}}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$

. . . *the* **gradient** *of the line*

Also:

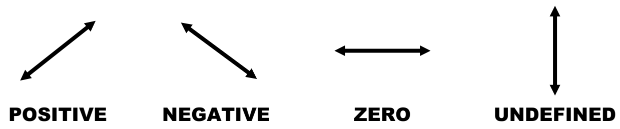


$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$

. . . *where θ is the* **angle of inclination** *of the line*

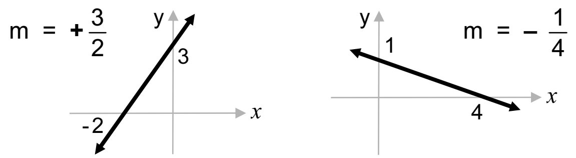
The Gradient of a line

Values



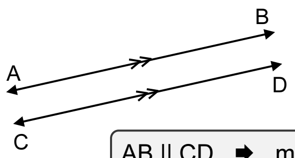
$m = +\frac{3}{2}$

$m = -\frac{1}{4}$



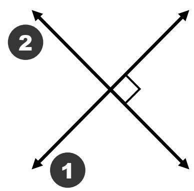
Parallel lines

Parallel lines have **equal** gradients.



AB || CD ➡ $m_{AB} = m_{CD}$

Perpendicular lines



If the gradient of line ❶ is $\frac{2}{3}$, then the gradient of line ❷ will be $-\frac{3}{2}$

Note: $m_1 \times m_2 = \left(+\frac{2}{3}\right)\left(-\frac{3}{2}\right) = -1$

i.e. The **product** of the gradients of ⊥ lines is **–1**.

Collinear points

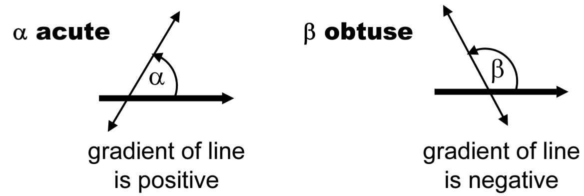
Three points A, B & C are collinear if the gradients of **A**B & **A**C are equal. (Note: Point **A** is common.)

$m_{AB} = m_{AC}$ ⬌ A, B & C are collinear

The Inclination of a line

Angles α and β below are **angles of inclination**.

The Inclination of a line is the **angle** which the line makes with the positive direction of the x-axis.



Gradient, **m = tan α** or **tan β** where α and β are the **∠^s of inclination**.

STRAIGHT LINE GRAPHS & their equations

Standard forms

- **y = mx + c:**

  where **m** = the gradient & **c** = the y-intercept

  When m = 0: **y = c** *… a line* || **x-axis**

  When c = 0: **y = mx** *… a line through the* **origin**

  Also: **x = k** *… a line* || **y-axis**

- **$y - y_1 = m(x - x_1)$:**

  where **m** = the gradient & **(x_1; y_1)** is a fixed point.

General form

The **general form** is **ax + by + c = 0**,

e.g. $2x + 3y + 6 = 0$

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3-in-1

Mathematics

FULL SOLUTIONS TO MODULE EXERCISES

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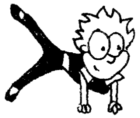
CONTENTS

MODULES

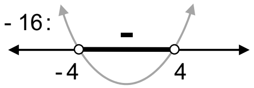


Examination Papers and Memos are in the Study Guide.

EXERCISE 3.5: Quadratic Inequalities

Questions on page 3.11

1. (a) $x^2 = 16$
$\therefore x^2 - 16 = 0$
$\therefore (x+4)(x-4) = 0$
$\therefore$ **$x = -4$ or 4** ◄
(The roots)

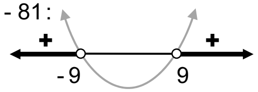
(b) $x^2 < 16$
$\therefore x^2 - 16 < 0$
$\therefore (x+4)(x-4) < 0$
$x^2 - 16$:


$\therefore$ **$-4 < x < 4$** ◄

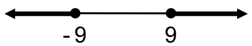
(c) $x^2 \le 16$
$\therefore x^2 - 16 \le 0$
$\therefore (x+4)(x-4) \le 0$
Combine (a) & (b):


$\therefore$ **$-4 \le x \le 4$** ◄

2. (a) $x^2 = 81$
$\therefore x^2 - 81 = 0$
$\therefore (x+9)(x-9) = 0$
$\therefore$ **$x = -9$ or 9** ◄
(The roots)

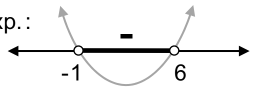
(b) $x^2 > 81$
$\therefore x^2 - 81 > 0$
$\therefore (x+9)(x-9) > 0$
$x^2 - 81$:


$\therefore$ **$x < -9$ or $x > 9$** ◄

(c) $x^2 \ge 81$
$\therefore x^2 - 81 \ge 0$
$\therefore (x+9)(x-9) \ge 0$
Combine (a) & (b):


$\therefore$ **$x \le -9$ or $x \ge 9$** ◄

3. (a) $x^2 - 5x - 6 = 0$
$\therefore (x+1)(x-6) = 0$
$\therefore$ **$x = -1$ or 6** ◄
(The roots)

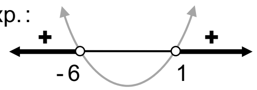
(b) $x^2 - 5x - 6 < 0$
$\therefore (x+1)(x-6) < 0$
Exp.:


$\therefore$ **$-1 < x < 6$** ◄

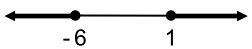
(c) $x^2 - 5x - 6 \le 0$
$\therefore (x+1)(x-6) \le 0$
Combine (a) & (b):


$\therefore$ **$-1 \le x \le 6$** ◄

4. (a) $x^2 + 5x - 6 = 0$
$\therefore (x+6)(x-1) = 0$
$\therefore$ **$x = -6$ or 1** ◄
(The roots)

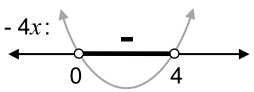
(b) $x^2 + 5x - 6 > 0$
$\therefore (x+6)(x-1) > 0$
Exp.:


$\therefore$ **$x < -6$ or $x > 1$** ◄

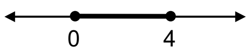
(c) $x^2 + 5x - 6 \ge 0$
$\therefore (x+6)(x-1) \ge 0$
Combine (a) & (b):


$\therefore$ **$x \le -6$ or $x \ge 1$** ◄

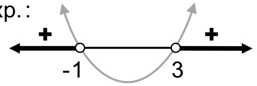
5. (a) $x^2 = 4x$
$\therefore x^2 - 4x = 0$
$\therefore x(x-4) = 0$
$\therefore$ **$x = 0$ or 4** ◄
(The roots)

(b) $x^2 < 4x$
$\therefore x^2 - 4x < 0$
$\therefore x(x-4) < 0$
$x^2 - 4x$:


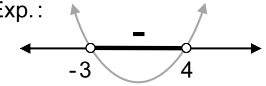
$\therefore$ **$0 < x < 4$** ◄

(c) $x^2 \le 4x$
$\therefore x^2 - 4x \le 0$
$\therefore x(x-4) \le 0$
Combine (a) & (b):


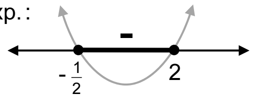
$\therefore$ **$0 \le x \le 4$** ◄

6. $x^2 - 2x - 3 > 0$
$\therefore (x-3)(x+1) > 0$
Exp.:


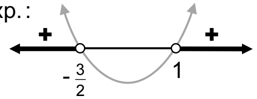
$\therefore$ **$x < -1$ or $x > 3$** ◄

7. $x^2 - x - 12 < 0$
$\therefore (x-4)(x+3) < 0$
Exp.:


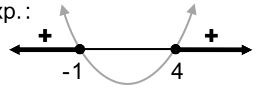
$\therefore$ **$-3 < x < 4$** ◄

8. $2x^2 - 3x - 2 \le 0$
$\therefore (2x+1)(x-2) \le 0$
Exp.:


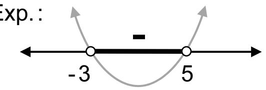
$\therefore$ **$-\frac{1}{2} \le x \le 2$** ◄

9. $3 - x < 2x^2$
$\therefore -2x^2 - x + 3 < 0$
$\times(-1)$ $\therefore 2x^2 + x - 3 > 0$
$\therefore (2x+3)(x-1) > 0$
Exp.:


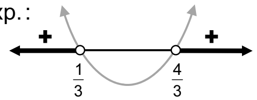
$\therefore$ **$x < -\frac{3}{2}$ or $x > 1$** ◄

10. $-x^2 + 3x + 4 \le 0$
$\times(-1)$ $\therefore x^2 - 3x - 4 \ge 0$
$\therefore (x+1)(x-4) \ge 0$
Exp.:


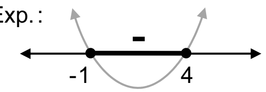
$\therefore$ **$x \le -1$ or $x \ge 4$** ◄

11. $x^2 - 2x - 15 < 0$
$\therefore (x+3)(x-5) < 0$
Exp.:


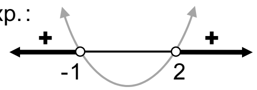
$\therefore$ **$-3 < x < 5$** ◄

12. $9x^2 - 12x + 4 > 3x$
$\therefore 9x^2 - 15x + 4 > 0$
$\therefore (3x-4)(3x-1) > 0$
Exp.:


$\therefore$ **$x < \frac{1}{3}$ or $x > \frac{4}{3}$** ◄

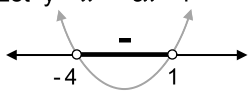
13. $x^2 - 3x + 2 \le 6$
$\therefore x^2 - 3x - 4 \le 0$
$\therefore (x-4)(x+1) \le 0$
Exp.:


$\therefore$ **$-1 \le x \le 4$** ◄

14. $-3(x+1)(x-2) < 0$
$\div(-3)$: $\therefore (x+1)(x-2) > 0$
Exp.:


$\therefore$ **$x < -1$ or $x > 2$** ◄

15. (a) $(x+3)(x-1) = -x + 1$
$\therefore x^2 + 2x - 3 = -x + 1$
$\therefore x^2 + 3x - 4 = 0$
$\therefore (x+4)(x-1) = 0$
$\therefore$ **$x = -4$ or $x = 1$** ◄

(b) Let $y = x^2 + 3x - 4$


$y < 0$ if **$-4 < x < 1$** ◄

1.3 $\mathbf{m_{PR} = -\frac{2}{4} = -\frac{1}{2}}$ ◄

1.4 $\theta = 180° - 26{,}57°$

. . . $\tan^{-1}\left(\frac{1}{2}\right) = 26{,}57°$

$\mathbf{= 153{,}43°}$ ◄

OR: $\theta = \tan^{-1}\left(-\frac{1}{2}\right) + 180°$
$= -26{,}57° + 180°$
$= 153{,}43°$

1.5 $\mathbf{y = -\frac{1}{2}x + 2}$ ◄ . . . *$m = -\frac{1}{2}$ and $c = 2$ in $y = mx + c$*

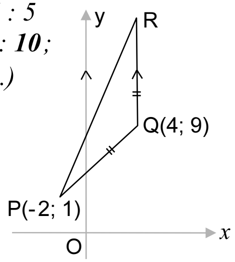
2.1 $PQ = \sqrt{(4+2)^2 + (9-1)^2}$ $\quad$ $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{6^2 + 8^2}$

$= \mathbf{10\ units}$ ◄ . . . *(3 : 4 : 5 = 6 : 8 : **10**; Pyth.)*

2.2 **R(4; 19)** ◄

QR = PQ = 10;
RQ || y-axis ➡ $x_R = x_Q$



3.1 $\mathbf{m_{AC} = \frac{1-2}{1-(-1)} = \frac{-1}{2} = -\frac{1}{2}}$ ◄

3.2 $m_{BC} = 2$ . . . $m_{BC} = -\frac{1}{m_{AC}}$

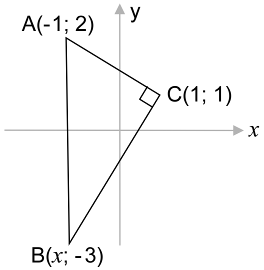
$\because BC \perp AC$

$\therefore \frac{1-(-3)}{1-x} = 2$

$\therefore 4 = 2 - 2x$

$\therefore 2x = -2$

$\therefore \mathbf{x = -1}$ ◄



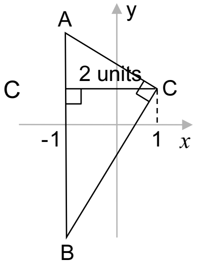
3.3 **AB** = 2 - (-3) **= 5 units** ◄ . . . $x_B = x_A$ ➡ $AB = y_A - y_B$
. . . *a vertical length*

3.4 Area of $\triangle$ABC

$= \frac{1}{2}AB \times$ height from C

$= \frac{1}{2}(5)(2)$

$= \mathbf{5\ units^2}$ ◄



The height is a horizontal length
$\because$ it is || x-axis.

4.1 y = 2x + 3 has gradient = 2

& 2y - kx = 16 ➡ 2y = kx + 16 ➡ $y = \frac{k}{2}x + 8$

$\therefore$ has gradient $= \frac{k}{2}$

$\therefore \frac{k}{2} = 2$ . . . *parallel lines have equal gradients*

$\therefore$ **k = 4** ◄

4.2 $m_{PR} = m_{PQ}$ . . . *P, Q & R are collinear*

$\frac{k-(-3)}{k-1} = \frac{6-(-3)}{4-1}$

$\therefore \frac{k+3}{k-1} = \frac{9}{3}$ (= 3)

$\therefore k + 3 = 3k - 3$

$\therefore -2k = -6$

$\therefore$ **k = 3** ◄

5.1 $\frac{k+4}{2} = 3$ . . . *M midpoint BC*

$\therefore k + 4 = 6$

$\therefore$ **k = 2** ◄

5.2.1 Point R is $\left(\frac{3+7}{2}; \frac{6+4}{2}\right)$

$\therefore$ R(5; 5)

$\therefore MR = \sqrt{(5-4)^2 + (5-3)^2}$

$= \sqrt{1+4}$

$= \sqrt{5}$

& $BA = \sqrt{(3-1)^2 + (6-2)^2}$

$= \sqrt{4+16}$

$= \sqrt{20}$

$= 2\sqrt{5}$ . . . $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4}\sqrt{5}$

$\therefore \mathbf{MR = \frac{1}{2}BA}$ ◄

5.2.2 Gradient of MR $= \frac{5-3}{5-4} = \frac{2}{1} = 2$

& Gradient of BA $= \frac{6-2}{3-1} = \frac{4}{2} = 2$

$\therefore$ **MR || BA** ◄ . . . *$m_{MR} = m_{BA}$*

6.1 ▶ M(0; p) is the midpoint of AC . . . *diagonals of a $||^m$ bisect one another*

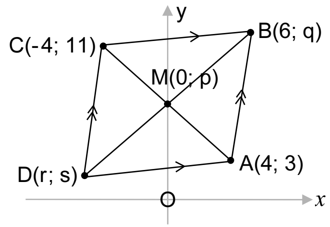
$\therefore \mathbf{p = \frac{11+3}{2} = 7}$ ◄

▶ B(6; q) on AB:

y = 5x - 17

$\therefore$ q = 5(6) - 17

$\therefore$ **q = 13** ◄



▶ M(0; 7) midpoint of DB where D(r; s) & B(6; 13)

➡ **r = -6** ◄ & **s = 1** ◄

6.2 $m_{CA} = \frac{3-11}{4-(-4)} = \frac{-8}{8} = -1$

& $m_{DB} = \frac{13-1}{6-(-6)} = \frac{12}{12} = 1$

$m_{CA} \times m_{DB} = -1$

$\therefore$ **CA $\perp$ DB** ◄

6.3 **$||^m$ ABCD is a rhombus** ◄ . . . *the diagonals cut at right angles.*

7.1 **Gradient OF** = tan(90° + 70°) = tan 160° = **-0,4** ◄

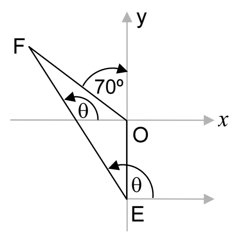
7.2 $\tan\theta = m_{EF} = -\frac{3}{2}$

$\therefore \theta = 180° - 56{,}3°$ . . . *ref. $\angle = \tan^{-1}1{,}5 = 56{,}3°$*

$= 123{,}7°$

$\therefore O\hat{E}F = 123{,}7° - 90°$

$\mathbf{= 33{,}7°}$ ◄



8.1.1 $m_{NK} = \frac{6-(-2)}{3-(-1)}$ . . . $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{8}{4}$

$= \mathbf{2}$ ◄

8.1.2 $\mathbf{m_{PK} = -\frac{1}{2}}$ ◄

EXERCISE 6.1: Parabolas

Questions on page 6.13

1.1 (a) (-3; -5) (b) $y = x^2 + 6x + 4$ (c) (0; 4)

1.2 (a) (1; 4) (b) $y = -x^2 + 2x + 3$ (c) (0; 3)

1.3 (a) (-2; 1) (b) $y = 2x^2 + 8x + 9$ (c) (0; 9)

2.1 (a) $y = x^2 + 8x + 10$ . . . *std form*

$\therefore\ y = x^2 + 8x + \mathbf{4^2} + 10 \mathbf{- 16}$

$\therefore\ y = (x + 4)^2 - 6$ . . . *t.pt. form*

(b) 4 units left and 6 units down

(c) turning point: (-4; -6)

(d) y-int: (0; 10) . . . *see the std form*

> **The y-intercept**
> *Either*
> - read it from the standard form
> *or*
> - substitute $x = 0$ in the t.p. form

2.2 (a) $y = x^2 - 6x + 11$

$\therefore\ y = x^2 - 6x \mathbf{+ 3^2} + 11 \mathbf{- 9}$

$\therefore\ y = (x - 3)^2 + 2$

(b) 3 units right and 2 units up

(c) turning point: (3; 2)

(d) y-intercept: (0; 11) . . . *see the standard form*

2.3 (a) $y = -x^2 + 4x + 5$

$\therefore\ y = -(x^2 - 4x - 5)$

$\therefore\ y = -(x^2 - 4x \mathbf{+ 2^2} - 5 \mathbf{- 4})$

$\therefore\ y = -[(x - 2)^2 - 9]$

$\therefore\ y = -(x - 2)^2 + 9$

(b) 2 units right and 9 units up

(c) turning point: (2; 9)

(d) y-intercept: (0; 5) . . . *see the standard form*

3.1 (a) $y = x^2 - 7x + 12$

$\therefore\ \mathbf{y = (x - 3)(x - 4)}$ ◄

(b) $\mathbf{x = 3}$ and $\mathbf{x = 4}$ ◄

(c) $\mathbf{x = 3\frac{1}{2}}$ ◄

3.2 (a) $y = -(x^2 - 7x + 12)$

$\therefore\ \mathbf{y = -(x - 3)(x - 4)}$ ◄

(b) $\mathbf{x = 3}$ & $\mathbf{x = 4}$ ◄

(c) $\mathbf{x = 3\frac{1}{2}}$ ◄

3.3 (a) $y = (x - 1)^2 - 4$

$\therefore\ y = x^2 - 2x - 3$

$\therefore\ \mathbf{y = (x + 1)(x - 3)}$ ◄

(b) $\mathbf{x = -1}$ and $\mathbf{x = 3}$ ◄

(c) $\mathbf{x = \frac{-1 + 3}{2} = 1}$ ◄

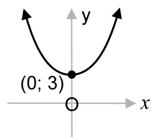
---

4. (a) $\mathbf{y = (x - 3)^2 - 1}$ ◄

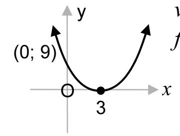
(b) $\mathbf{y = (x + 2)^2 + 5}$ ◄

> *Note: a = 1, given*

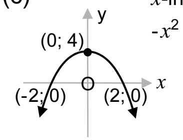
5.1 (a)



(b) x-int: x = 3 . . . *the 'zero value' of x for ( )² = 0*



(c)



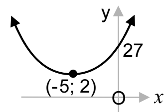
x-ints:

$-x^2 + 4 = 0$

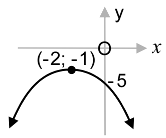
$x^2 = 4$

$x = \pm 2$

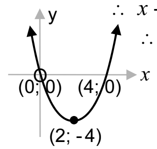
(d)



(e)



(f)



x-ints:

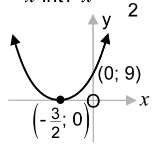
$(x - 2)^2 = 4$

$\therefore\ x - 2 = \pm 2$

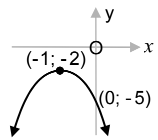
$\therefore\ x = 2 \pm 2$

$= 0$ or 4

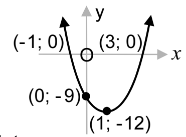
(g) x-int: $x = -\frac{3}{2}$ . . . *the 'zero value' of x for ( )² = 0*



(h)



(i)



x-ints:

$3(x - 1)^2 - 12 = 0$

$3(x - 1)^2 = 12$

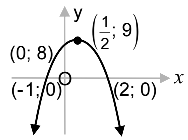
$(x - 1)^2 = 4$

$x - 1 = \pm 2$

$x = 1 \pm 2$

$\therefore\ x = -1$ or 3

(j)



x-ints:

$-(2x - 1)^2 + 9 = 0$

$(2x - 1)^2 = 9$

$2x - 1 = \pm 3$

$2x = 1 \pm 3$

$x = \frac{1 \pm 3}{2}$

$\therefore\ x = -1$ or 2

5.2.1 (a) 3 units down

(b) 3 units left

(d) 5 units right and 2 units down

(f) 2 units left and 4 units up

5.2.2 (c) 4 units down

(e) 2 units right and 1 unit up

6.1

> (a) → (c): Use $\mathbf{y = ax^2 + q}$
> *These graphs are symmetrical about the y-axis.*

(a) $y = ax^2 + 3$

(3; 6): $6 = a(3)^2 + 3$

$\therefore\ 3 = 9a$

$\therefore\ a = \frac{1}{3}$

$\therefore$ Eqn: $\mathbf{y = \frac{1}{3}x^2 + 3}$ ◄

(b) $y = ax^2 + 4$

(-4; 0): $0 = a(-4)^2 + 4$

$\therefore\ -4 = 16a$

$\therefore\ a = -\frac{1}{4}$

$\therefore$ Eqn: $\mathbf{y = -\frac{1}{4}x^2 + 4}$ ◄

(c) $y = ax^2$ . . . **c = 0**

(-3; 4): $4 = a(-3)^2$

$\therefore\ 4 = 9a$

$\therefore\ a = \frac{4}{9}$

$\therefore$ Eqn: $\mathbf{y = \frac{4}{9}x^2}$ ◄

> (d) → (f): Use $\mathbf{y = a(x - p)^2 + q}$
> *These graphs are not symmetrical about the y-axis.*

(d) t.p.: (2; 0)

$\therefore\ y = a(x - 2)^2$

(0; 2): $\therefore\ 2 = a(-2)^2$

$\therefore\ a = \frac{1}{2}$

$\therefore$ Eqn: $\mathbf{y = \frac{1}{2}(x - 2)^2}$ ◄

(e) t.p.: (1; 2)

$\therefore\ y = a(x - 1)^2 + 2$

(0; 5): $\therefore\ 5 = a(-1)^2 + 2$

$\therefore\ a = 3$

$\therefore$ Eqn: $\mathbf{y = 3(x - 1)^2 + 2}$ ◄

(f) Symmetry $x = 2$ & Maximum y = 9

$\therefore\ y = a(x - 2)^2 + 9$

(-1; 0): $\therefore\ 0 = a(-1 - 2)^2 + 9$

$\therefore\ 9a = -9$

$\therefore\ a = -1$

$\therefore$ Eqn: $\mathbf{y = -(x - 2)^2 + 9}$ ◄

(b) $\frac{\sin\theta}{\sin\theta} - (-\sin\theta)(-\sin\theta)(1) = 1 - \sin^2\theta$

$= \mathbf{\cos^2\theta}$ ◀

(c) $\cos x + (-\cos x) + (-1)(+\tan 45^\circ) + (-\cos x)^2$

$= -1\,(+1) + \cos^2 x$

$= -1 + \cos^2 x$

$= -(1 - \cos^2 x)$

$= \mathbf{-\sin^2 x}$ ◀

(d) $\frac{\cos A + \cos A}{\cos A}$

$= \frac{2\cos A}{\cos A}$

$= \mathbf{2}$ ◀

(e) $\frac{\sin\theta\,.\,\tan\theta\,.\cos\theta}{(-\tan\theta)\,.\,(-\sin\theta)}$

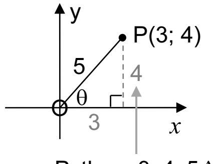
$= \mathbf{\cos\theta}$ ◀

4. (a) $5\cos\theta - 3 = 0$

$\therefore\ 5\cos\theta = 3$

$\therefore\ \cos\theta = \frac{3}{5}$

$\therefore\ \mathbf{\sin\theta = \frac{4}{5}}$ ◀

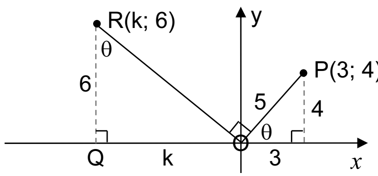


Pythag; 3 : 4 : 5∆

(b) The lengths $\frac{6}{OQ} = \frac{3}{4}$ . . . *see the position of θ in ∆ROQ*

$OQ = 8$ units

$\therefore\ \mathbf{k = -8}$ ◀



5. (a) $\cos(90^\circ + x).\sin(360^\circ - x) + \sin^2(90^\circ - x)$

$= (-\sin x).(-\sin x) + \cos^2 x$

$= \sin^2 x + \cos^2 x$

$= \mathbf{1}$ ◀

(b) $\frac{\sin(180^\circ - x)\,.\,\sin(90^\circ + x)}{\cos^2(180^\circ + x)}$

$= \frac{\sin x\,.\cos x}{(-\cos x)^2} = \frac{\sin x\,.\cos x}{\cos^2 x} = \frac{\sin x}{\cos x} = \mathbf{\tan x}$ ◀

EXERCISE 7.4: All Standard General Forms

Questions on page 7.16

1. (a) $-\sin 20^\circ$ (b) $-\cos 50^\circ$ (c) $\tan 70^\circ$ (d) $\cos 50^\circ$
(e) $\sin 40^\circ$ (f) $-\tan 80^\circ$ (g) $\cos 40^\circ$ (h) $-\sin 25^\circ$
(i) $-\cos 20^\circ$ (j) $-\tan 70^\circ$ (k) $-\sin 40^\circ$ (l) $\tan 80^\circ$

2. (a) $\frac{\sin 40^\circ.\tan 45^\circ}{-\cos 50^\circ.\tan 30^\circ}$

$= \frac{\cos 50^\circ.1}{-\cos 50^\circ.\frac{1}{\sqrt{3}}}$

$= \mathbf{-\sqrt{3}}$ ◀

(b) $\frac{\cos 60^\circ.(\tan 30^\circ)^2}{(-\sin 30^\circ).(-\cos 60^\circ)}$

$= \frac{\frac{1}{2}.\left(\frac{1}{\sqrt{3}}\right)^2}{\left(-\frac{1}{2}\right).\left(-\frac{1}{2}\right)}$

$= \frac{\frac{1}{2}.\frac{1}{3}}{\frac{1}{4}}$

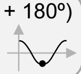
$= \mathbf{\frac{2}{3}}$ ◀

(c) $\frac{\sin 50^\circ.\sqrt{3}}{{}^*\cos 180^\circ.\tan 50^\circ.\sin 40^\circ}$

$= \frac{\sin 50^\circ.\sqrt{3}}{(-1).\frac{\sin 50^\circ}{\cos 50^\circ}.\cos 50^\circ}$

$= \mathbf{-\sqrt{3}}$ ◀

* $\cos 540^\circ = \cos(360^\circ + 180^\circ)$
$= \cos 180^\circ$
$= -1$



(d) $\frac{\tan 60^\circ}{-\cos 30^\circ} - \frac{-\cos 20^\circ}{-\sin 70^\circ}$

$= \frac{\sqrt{3}}{-\frac{\sqrt{3}}{2}} - \frac{\cos 20^\circ}{\cos 20^\circ}$

$= -2 - 1$

$= \mathbf{-3}$ ◀

3. (a) $\frac{(-\sin x).\cos x.(-\cos 60^\circ)}{\tan x}$

$= \frac{(-\sin x).\cos x.\left(-\frac{1}{2}\right)}{\frac{\sin x}{\cos x}}$

$= \mathbf{\frac{1}{2}\cos^2 x}$ ◀

(b) $\frac{(-\cos x)(-\cos x) - 1}{(\tan^2 x)(\cos x)(\cos x)}$

$= \frac{\cos^2 x - 1}{\frac{\sin^2 x}{\cos^2 x} \times \frac{\cos^2 x}{1}} = \frac{-(1 - \cos^2 x)}{\sin^2 x} = \frac{-\sin^2 x}{\sin^2 x} = \mathbf{-1}$ ◀

(c) $(+\cos x).(-\cos x) + (-\tan x).\cos x.\sin x$

$= -\cos^2 x - \frac{\sin x}{\cos x}.\cos x.\sin x$

$= -(\cos^2 x + \sin^2 x)$

$= \mathbf{-1}$ ◀

EXERCISE 7.5: Identities

Questions on page 7.16

1. (a) $\frac{\sin x}{\cos x} \times \cos x$

$= \mathbf{\sin x}$ ◀

(b) $\frac{\cos^2 x}{\cos^2 x}$

$= \mathbf{1}$ ◀

(c) $\frac{1 - \cos^2\theta}{\sin\theta}$

$= \frac{\sin^2\theta}{\sin\theta}$

$= \mathbf{\sin\theta}$ ◀

(d) $\frac{1 - \cos^2\theta}{\cos^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta}$

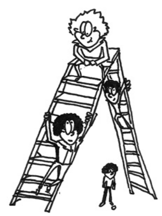
$= \mathbf{\tan^2\theta}$ ◀

2. (a) **LHS** $= \frac{\sin\theta(\sin^2\theta + \cos^2\theta)}{\cos\theta}$

$= \frac{\sin\theta(1)}{\cos\theta}$

$= \tan\theta$

$=$ **RHS** ◀



(b) **LHS** $= \frac{\cos\theta}{1 + \sin\theta} + \frac{1 + \sin\theta}{\cos\theta}$

$= \frac{\cos^2\theta + (1 + \sin\theta)^2}{\cos\theta(1 + \sin\theta)}$

$= \frac{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}{\cos\theta(1 + \sin\theta)}$

$= \frac{2 + 2\sin\theta}{\cos\theta(1 + \sin\theta)}$ . . . $\sin^2\theta + \cos^2\theta = 1$

$= \frac{2(1 + \sin\theta)}{\cos\theta(1 + \sin\theta)}$

$= \frac{2}{\cos\theta} =$ **RHS** ◀

18.2 $P\hat{T}R = Q\hat{P}S + \hat{Q}$ . . . *exterior ∠ of ΔPQT*
$= 2x$

$\therefore P\hat{T}S = 2\hat{Q}$

18.3 $P\hat{T}R = P\hat{O}R$ . . . *both = 2x*

∴ P, O, T and R are concyclic

. . . *PR subtends equal ∠^s at T and O, converse ∠^s in same segment*

EXERCISE 9.5: Tangents

Questions on page 9.21

1.1 $O\hat{P}A = 90°$ . . . *tan ⊥ rad*
∴ **x = 50°** ◄ . . . *∠ sum of Δ*

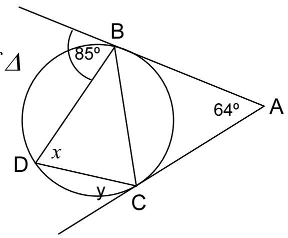
$O\hat{P}B = 90°$ . . . *tan ⊥ rad*
∴ **y = 30°** ◄ . . . *∠ sum of Δ*

1.2 **x = 50°** ◄ ; **y = 70°** ◄ . . . *tan chord theorem*

1.3 $A\hat{B}C = A\hat{C}B$ . . . *tans from a common point A*

$= \frac{1}{2}(180° - 64°)$ . . . *sum of ∠^s of Δ*

$= 58°$



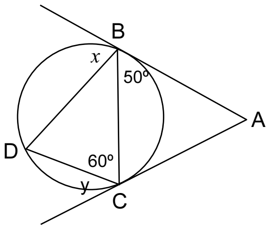
∴ **x = 58°** ◄
. . . *tan chord theorem*

$\therefore D\hat{B}C = 180° - (85° + 58°)$ . . . *∠^s on a str. line*
$= 37°$

∴ **y = 37°** ◄ . . . *tan chord theorem*

1.4 **x = 60°** ◄
. . . *tan chord theorem*

$\hat{D} = 50°$
. . . *tan chord theorem*



$\therefore D\hat{B}C = 70°$ . . . *sum of ∠^s of ΔDBC*

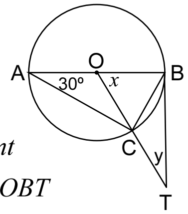
∴ **y = 70°** ◄ . . . *tan chord theorem*

1.5 $O\hat{C}A = 30°$ . . . *∠^s opp = radii*

∴ **x = 60°** ◄ . . . *ext. ∠ of Δ*
or ∠ at centre = 2 × ∠ at circum.

$O\hat{B}T = 90°$ . . . *radius ⊥ tangent*

∴ **y = 30°** ◄ . . . *sum of ∠^s of ΔOBT*

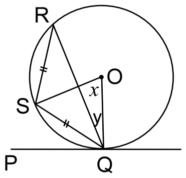


1.6 $\hat{R} = S\hat{Q}R$ . . . *∠^s opp = sides*

$= \frac{1}{2}(180° - 100°)$ . . . *∠ sum of Δ*

$= 40°$

∴ x = 2(40°) . . . *∠ at centre = 2 × ∠ at circum.*
= 80° ◄



$\therefore O\hat{Q}S = O\hat{S}Q$ . . . *∠^s opp = radii*

$= \frac{1}{2}(180° - 80°)$ . . . *∠ sum of Δ*

$= 50°$

But $R\hat{Q}S = 40°$

∴ **y** = 50° - 40° = **10°** ◄

OR:

For y . . .

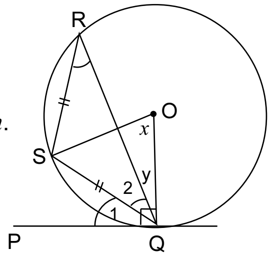
$O\hat{Q}P = 90°$ . . . *tan ⊥ rad*

$\hat{Q}_1 = \hat{R}$ . . . *tan chord thm.*

$= \hat{Q}_2$ . . . *proved above*

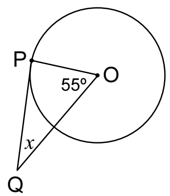
$= 40°$

∴ **y** = 90° - 2(40°) = **10°** ◄



2.1 $O\hat{P}Q = 90°$ . . . *radius ⊥ tangent*

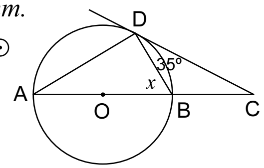
∴ x = 90° - 55° . . . *sum of ∠^s of Δ*
= 35° ◄



2.2 $\hat{A} = 35°$ . . . *tan chord thm.*

$A\hat{D}B = 90°$ . . . *∠ in semi-⊙*

∴ x = 90° - 35° . . . *sum of ∠^s of Δ*
= 55° ◄



2.3 $O\hat{P}Q = 90°$ . . . *radius ⊥ tangent*

∴ **x = 8 cm** ◄ . . . *3 : 4 : 5 = 6 : **8** : 10 ; Thm. of Pythag.*

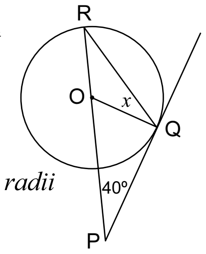
2.4 $O\hat{Q}P = 90°$ . . . *radius ⊥ tangent*

$\therefore P\hat{O}Q = 50°$ . . . *sum of ∠^s of Δ*

But $P\hat{O}Q = \hat{R} + x$ . . . *ext. ∠ of Δ*

$= 2x$ . . . *$\hat{R} = x$; ∠^s opp = radii*

∴ **x = 25°** ◄

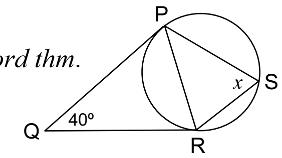


2.5 $Q\hat{P}R = Q\hat{R}P$ . . . *tangents from a common pt.*

$= \frac{1}{2}(180° - 40°)$ . . . *∠ sum of Δ*

$= 70°$

∴ **x = 70°** ◄ . . . *tan chord thm.*



2.6 $O\hat{P}Q = O\hat{R}Q = 90°$ . . . *radii ⊥ tangents*

∴ PQRO is a cyclic quad. . . . *opp. ∠^s are suppl.*

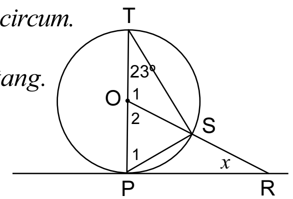
∴ x = 180° - 50° . . . *opp. ∠^s of c.q.*
= 30° ◄

OR: x = 360° - 2(90°) - 150° . . . *∠ sum of a quad.*
= 30° ◄

2.7 $\hat{O}_2 = 2(23°)$ . . . *∠ at centre = 2 × ∠ at circum.*
$= 46°$

$O\hat{P}R = 90°$ . . . *diam. ⊥ tang.*

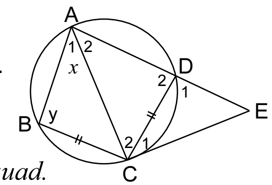
∴ **x = 44°** ◄
. . . *∠ sum of ΔOPR*



3.1 (a) **$\hat{A}_2 = x$** ◄ . . . *equal chords ; equal ∠^s*

∴ $\hat{C}_1 = x$ ◄
. . . *tan chord thm.*

(b) **$\hat{D}_1 = y$** ◄
. . . *ext. ∠ of cyclic quad.*



12.6 The area of quadrilateral KPNQ

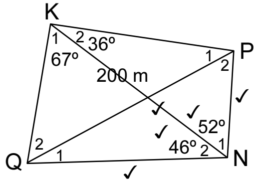
= the area of ΔKPN + the area of ΔKQN

$= \frac{1}{2}$ PN.KN.sin 52º + $\frac{1}{2}$ QN.KN.sin 46º

. . . *where QN = KN = 200 m & PN = 118 m*

= 9 298,526... + 14 386,796...

≃ **23 685,32 m²** ◄



13.1 M$\hat{P}$N = 180º - (52º + 32º) . . . *∠ sum of ΔMPN*

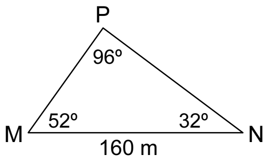
= 96º ◄

13.2 In ΔMPN:

$\frac{\textbf{MP}}{\sin 32^\circ} = \frac{160}{\sin 96^\circ}$

∴ MP = $\frac{160 \sin 32^\circ}{\sin 96^\circ}$

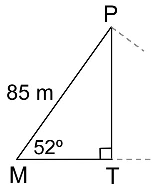
≃ **85 m** ◄



13.3 In ΔPMT: $\frac{\textbf{PT}}{85} = \sin 52^\circ$

∴ PT = 85 sin 52º

≃ **67 m** ◄



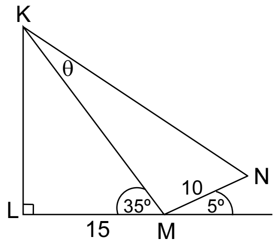
14.1 $c^2 = a^2 + b^2 - 2ab \cos C$

14.2.1 In ΔKLM:

$\frac{15}{KM} = \cos 35^\circ$

∴ $\frac{15}{\cos 35^\circ} = KM$

∴ **KM** ≃ **18,3 m** ◄



14.2.2 In ΔKMN: K$\hat{M}$N = 140º . . . *∠ sum of Δ*

& $KN^2 = 18{,}3^2 + 10^2 - 2(18{,}3)(10) \cos 140^\circ$

= 715,26...

∴ **KN** ≃ **26,7 m** ◄

14.2.3 In ΔKMN: ➡ $\frac{\textbf{sin}\ \boldsymbol{\theta}}{10} = \frac{\sin 140^\circ}{26{,}7}$

Place θ in the TOP LEFT position.

∴ $\sin\theta = \frac{10 \sin 140^\circ}{26{,}7}$

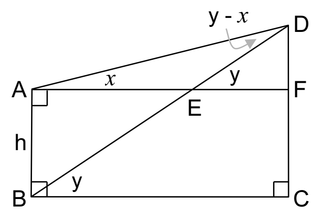
= 0,2407...

∴ θ ≃ **13,9º** ◄

Note: θ is acute ∵ already K$\hat{M}$N is obtuse.

15.1 $\frac{\textbf{sin A}}{\textbf{a}} = \frac{\textbf{sin B}}{\textbf{b}} = \frac{\textbf{sin C}}{\textbf{c}}$ ◄

15.2



15.2.1 D$\hat{E}$F = y . . . *corresponding ∠s ; AF || BC*

∴ A$\hat{D}$B = y - x . . . *exterior ∠ of ΔDAE*

& D$\hat{A}$B = 90º + x

In ΔDAB: $\frac{BD}{\sin(90^\circ + x)} = \frac{h}{\sin(y - x)}$ . . . *formula in 15.1*

∴ $BD = \frac{h \sin(90^\circ + x)}{\sin(y - x)}$

∴ $\textbf{BD} = \frac{\textbf{h cos}\ \boldsymbol{x}}{\textbf{sin(y -}\ \boldsymbol{x}\textbf{)}}$ ◄

15.2.2 $BD = \frac{8 \cos 31^\circ}{\sin(61^\circ - 31^\circ)}$

∴ $BD = \frac{8 \cos 31^\circ}{\sin 30^\circ}$

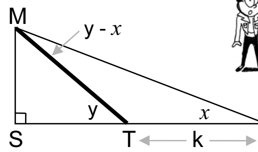
∴ BD = 13,71. . . m

In ΔDBC: $\frac{CD}{BD} = \sin y$

∴ CD = BD sin 61º

∴ **CD = 12 m** ◄

16.



For Questions 16 and 17: Be sure to read the '2D Proofs: steps in the process' on page 10.13.

16.1 **Exterior ∠ of Δ = sum of the two int. opp. ∠s** ◄

16.2 In ΔMTN: $\frac{\textbf{MT}}{\sin x} = \frac{k}{\sin(y - x)}$

∴ $MT = \frac{k \sin x}{\sin(y - x)}$ . . . ❶

It is important to place the required side in the **TOP LEFT** position.

16.3 In right ∠d ΔMST: $\frac{\textbf{MS}}{MT} = \sin y$

∴ MS = (MT) sin y . . . ❷

❶ in ❷: ∴ $MS = \frac{k \sin x . \sin y}{\sin(y - x)}$ ◄

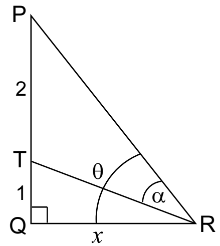
Note: MT is the **LINK** between the 2 Δs: rt ∠d ΔMST and non-rt ∠d ΔMTN

17.1 In ΔTQR: T$\hat{R}$Q = θ - α

& $\frac{x}{TR} = \cos(\theta - \alpha)$

∴ $x = TR \cos(\theta - \alpha)$

∴ $\textbf{TR} = \frac{\boldsymbol{x}}{\textbf{cos}(\boldsymbol{\theta - \alpha})}$ ◄



17.2 $\hat{P}$ **= 90º - θ** ◄ . . . *∠ sum of Δ*

17.3 In ΔPTR: $\frac{TR}{\sin(90^\circ - \theta)} = \frac{2}{\sin\alpha}$

∴ $\textbf{TR} = \frac{\textbf{2 cos}\ \boldsymbol{\theta}}{\textbf{sin}\ \boldsymbol{\alpha}}$ ◄

17.4 ∴ $\frac{x}{\cos(\theta - \alpha)} = \frac{2 \cos\theta}{\sin\alpha}$ . . . *both = TR*

∴ $\boldsymbol{x} = \frac{\textbf{2 cos}\ \boldsymbol{\theta}\textbf{.cos}(\boldsymbol{\theta - \alpha})}{\textbf{sin}\ \boldsymbol{\alpha}}$ ◄

17.5 $\boldsymbol{x} = \frac{\textbf{2 cos 50º.cos(50º - 30º)}}{\textbf{sin 30º}}$ ≃ **2,4 metres** ◄