Varsity Mathematics Preparation

John Webb

GRADE

12

CAPS

THE ANSWER SERIES *Your Key to Exam Success*

Varsity Maths Prep

CLASS TEXT & STUDY GUIDE

The Answer Series Maths University Preparation study guide is an essential resource if you're serious about an academic, maths-oriented future.

Emeritus Professor John Webb compiled this book to help prospective university learners negotiate the dire challenges they face in Mathematics, as well as courses in Science, Engineering and Business Science.

This self-study guide allows you to develop in areas such as:

- Algebraic expertise
- Trigonometry skills
- Problem-solving
- Geometric insight
- Numerical facility
- Logical reasoning
- Flexible thinking

These skills cannot be taught. They are best achieved without assistance, prior to entering university.

The problem-solving techniques which learners could acquire from dedicated, independent use of this outstanding booklet will contribute significantly to their success in the National Benchmark tests (NBTs).

THE ANSWER SERIES *Your Key to Exam Success*

VARSITY MATHS PREP

Skills you will need
to succeed in Maths 101

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CONTENTS

HOW TO USE THIS BOOK

- Start with the Basic Arithmetic, Basic Algebra and Basic Geometry problem sets. The later sets are a little harder, and may contain problems requiring ideas you will have gained from the earlier sets.
- Give yourself at least an hour or two to tackle a problem set, doing the problems in one continuous concentrated session.
- Don't allow yourself to be distracted by tweets or emails. Switch off all mobile devices!
- You don't have to do the problems in order. If a problem looks complicated, look again. It may have a simple solution if viewed from a different perspective. If you are still baffled, don't give up quickly. Come back to it later.
- No diagrams have been given. That is deliberate. It is a useful skill to be able to draw a figure from a written description.
- No calculators! You should be able to do simple arithmetic in your head, and none of the problems requires more than pencil-and-paper calculations.
- No formula sheet! You must have the standard trig and algebra formulas at your fingertips.
- These problems are for you to do by yourself, and certainly not with an "extra lessons tutor". However, working through them with a friend could be useful.
- When you have finished a problem set, check your answers for quick feedback. Before looking at the solutions, go back to any problem you got wrong and see if you can find your mistake. If you can't, look at the full solution.
- Even if you have got a full house of correct answers, read the solutions carefully. You may have got the right answer by luck, or by using a wrong method. The solutions may also give you alternative approaches, quicker methods and extra insights into the problems.
- Every wrong answer indicates a possible weakness in your mathematical background that needs to be fixed before your first Maths I lecture at your chosen university.

BASIC ARITHMETIC PROBLEMS

1. 728×125 is equal to

   (A) 91 000 (B) 92 600 (C) 78 500 (D) 93 000 (E) 89 500

   REMINDER: No calculators!

2. When written as a recurring decimal, $\dfrac{1}{7}$ is equal to

   (A) $0.153874\cdots$ (B) $0.142857\cdots$ (C) $0.123457\cdots$ (D) $0.148725\cdots$
   (E) $0.154278\cdots$

3. $\sqrt{1.44} + \sqrt{0.25}$ is equal to

   (A) 1.3 (B) 1.4 (C) 1.5 (D) 1.6 (E) 1.7

4. $\dfrac{\frac{1}{6} + \frac{1}{12}}{\frac{1}{20} + \frac{1}{30}}$ is equal to

   (A) $\frac{2}{3}$ (B) $\frac{3}{4}$ (C) 2 (D) $\frac{3}{2}$ (E) 3

5. 44% of 75 plus 38% of 50 is equal to

   (A) 50 (B) 52 (C) 54 (D) 56 (E) 58

6. Which of the following numbers is not a multiple of 7?

   (A) 78 428 (B) 91 770 (C) 14 784 (D) 21 764 (E) 35 504

7. $\sqrt[4]{0.0256}$ is equal to

   (A) 0.04 (B) 0.004 (C) 0.64 (D) 0.4 (E) 0.064

8. Which of the following statements is true?

   (A) $11^8 < 2^{28} < 5^{12}$

   (B) $2^{28} < 11^8 < 5^{12}$

   (C) $11^8 < 5^{12} < 2^{28}$

   (D) $5^{12} < 11^8 < 2^{28}$

   (E) $2^{28} < 5^{12} < 11^8$

9. Which of the following is a prime number?

   (A) 451 (B) 452 (3) 459 (D) 465 (E) 467

10. $\dfrac{2}{\sqrt{2}} + \dfrac{1}{\sqrt{2}+1} + (1-\sqrt{2})^2$ is equal to

    (A) $2+\sqrt{2}$ (B) 1 (C) $2-\sqrt{2}$ (D) 2 (E) $3-2\sqrt{2}$

BASIC ALGEBRA PROBLEMS

1. When $(3a-2b)(7b-5c)(6c-9a)$ is multiplied out, what is the coefficient of abc?

   (A) 40 (B) 36 (C) 44 (D) 49 (E) 52

2. $(a^{-1}+b^{-1})^{-2}$ is equal to

   (A) a^2+b^2 (B) $a^{-2}+b^{-2}$ (C) $\dfrac{a^2b^2}{(a+b)^2}$ (D) $\dfrac{a^2+b^2}{a^2b^2}$ (E) $2(a+b)$

3. The roots of the quadratic equation $3x^2-8x+3=0$ are

   (A) $\frac{1}{2}(6\pm\sqrt{7})$ (B) -3 and $\frac{1}{3}$ (C) $\frac{1}{3}(4\pm\sqrt{7})$ (D) 1 and $-\frac{1}{9}$
   (E) $\frac{1}{6}(4\pm\sqrt{14})$

4. Which of the following is not a factor of $6x^4+5x^3-75x^2+10x+24$?

   (A) $2x+1$ (B) $x-3$ (C) $x+4$ (D) $3x-2$ (E) $x-5$

5. $\dfrac{3}{x-2}-\dfrac{2}{x+3}$ is equal to

   (A) $\dfrac{x+13}{x^2+x-6}$ (B) $\dfrac{x-5}{x^2-x-6}$ (C) $\dfrac{2x+13}{x^2-x-6}$ (D) $\dfrac{x+13}{x^2-x+6}$

   (E) $\dfrac{x-8}{2x^2+x-3}$

6. When $(3x^2-2x+6)(x^2-4x+7)(x^2+3x-1)$ is multiplied out to the form $ax^6+bx^5+cx^4+dx^3+ex+f$, what is the value of $a+b+c+d+e+f$?

   (A) 84 (B) 96 (C) 72 (D) 108 (E) 120

7. The set of all real numbers x such that $x^2<5x+24$ is the interval

   (A) $(-3,8)$ (B) $(1,7)$ (C) $(3,8)$ (D) $(-8,-3)$ (E) $(-2,4)$

8. The sum $\sum_{n=1}^{\infty}(\frac{2}{3})^n$ is equal to

   (A) $\frac{3}{2}$ (B) 2 (C) $\frac{2}{3}$ (D) 3 (E) 6

9. If

$$2x+5y+4z=13$$
$$5x+4y+2z=15$$
$$4x+2y+5z=16$$

   then $x+y+z$ is equal to

   (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

BASIC GEOMETRY PROBLEMS

1. The points A, B, C and D lie in a straight line, in that order, and E is a point not on the line. If $\angle EBA = 130°$ and $\angle ECD = 94°$, then $\angle BEC$ is equal to

   (A) $44°$ (B) $58°$ (C) $61°$ (D) $63°$ (E) $67°$

2. The length of the hypotenuse of a right-angled isosceles triangle is a. What is the area of the triangle?

   (A) $\frac{1}{4}a^2$ (B) $a^2\sqrt{2}$ (C) $\frac{1}{2}a^2$ (D) $\frac{1}{\sqrt{2}}a^2$ (E) a^2

3. A circle cuts the x-axis at $(1, 0)$ and $(3, 0)$, and the y-axis at $(0, 1)$ and $(0, 3)$. What is the centre of the circle?

   (A) $(1, 3)$ (B) $(2, 2)$ (C) $(3, 1)$ (D) $(2, -2)$ (E) $(-2, 2)$

4. If A and B are two points in the plane, the set of all points P such that $AP = BP$ lie

   (A) on a straight line (B) on a parabola (C) on a hyperbola
   (D) on a circle (E) on an ellipse.

5. If $\sin A = \frac{1}{4}$, then $\sin 3A$ is equal to

   (A) $\frac{3}{4}$ (B) $\frac{1}{64}$ (C) $\frac{3}{8}$ (D) $\frac{11}{16}$ (E) $\frac{1}{12}$

6. A pyramid has a square base with sides of length 4 and the four sloping edges have length 5. What is the height of the pyramid?

   (A) $3\sqrt{2}$ (B) $2\sqrt{3}$ (C) 4 (D) $\sqrt{15}$ (E) $\sqrt{17}$

7. If $\cos 2x = \frac{1}{8}$ and $0° < x < 90°$, then $\cos x$ is equal to

   (A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{16}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

8. In triangle ABC, D, E and F lie on sides AB, BC and CA, respectively, so that $BD = BE$ and $CE = CF$. If $\angle A = 40°$, what is the size of $\angle DEF$?

   (A) $70°$ (B) $50°$ (C) $40°$ (D) $100°$ (E) $80°$

9. At what point do the straight lines $3x + 4y + 5 = 0$ and $2x - y - 4 = 0$ intersect?

   (A) $(-1, -3)$ (B) $(1, -2)$ (C) $(2, -3)$ (D) $(3, -1)$
   (E) The lines do not intersect.

PROBLEMS 5

1. The sides of quadrilateral $ABCD$ are produced:
AB is produced to E, BC is produced to F and CD is produced to G.
If $\angle EBC = 79°$, $\angle FCD = 64°$ and $\angle GDA = 127°$, then $\angle BAD$ is equal to
(A) $61°$ (B) $72°$ (C) $83°$ (D) $90°$ (E) $105°$

2. The graph of $y = (x-3)(1-x)$ is tangent to the graph of $y = kx^2$. Determine k.
(A) $\frac{2}{5}$ (B) $\frac{1}{3}$ (C) -2 (D) $\sqrt{2}$ (E) $-\frac{1}{2}$

3. The solution of the inequality $\log_2 x + \log_2(x-3) < 2$ is
(A) $-1 < x < 4$ (B) $x > 0$ (C) $0 < x < 4$ (D) $3 < x < 4$ (E) $x > 3$

4. The roots of the equation $x^2 - 2x - 7 = 0$ are a and b. Which of the following equations has roots $a+1$ and $b+1$?
(A) $x^2 - 3x - 8 = 0$ (B) $x^2 - x - 6 = 0$ (C) $x^2 - 4x - 4 = 0$
(D) $x^2 + 2x - 8 = 0$ (E) $x^2 + 2x - 5 = 0$.

5. In triangle ABC, with $AB = c$, $BC = a$ and $CA = b$,
$$\frac{4 \times \text{Area } ABC}{b^2 + c^2 - a^2}$$
is equal to
(A) $\cos A$ (B) $\tan A$ (C) $\sin 2A$ (D) $\frac{1}{2}\cos A$ (E) $2\sin A$

6. If $f(x) = 3x + 5$ and $g(x) = 4x + 7$, then $g(f(x)) - f(g(x))$ is equal to
(A) x (B) 0 (C) $x + 1$ (D) 1 (E) $x - 1$

7. In triangle ABC, D, E and F lie on sides AB, BC and CA, respectively, so that $BD = BE$ and $CE = CF$. If $\angle A = x$, what is the size of $\angle DEF$?
(A) $90° + x$ (B) $180° - 2x$ (C) $90° + 2x$ (D) $90° - \frac{1}{2}x$ (E) $180° - x$

8. Let f be a function defined by the equation $f(x) = 3^x$, where x is a real number. Which of the following is true for all real numbers a and b?
(A) $f(ab) = f(a) + f(b)$ (B) $f(ab) = f(a)f(b)$
(C) $f(ab) = 3f(ab-1)$ (D) $f(a+b) = f(a) + f(b)$
(E) $f(3ab) = f(ab+3)$

BASIC ARITHMETIC SOLUTIONS

1. $728 \times 125 = 728 \times \frac{1}{8}(1000) = \frac{1}{8}(728\ 000) = 91\ 000$ **(A)**

2. $\dfrac{1}{7} = 0.142857\cdots$ **(B)**

3. $\sqrt{1.44} + \sqrt{0.25} = 1.2 + 0.5 = 1.7$ **(E)**

4. $\dfrac{\frac{1}{6}+\frac{1}{12}}{\frac{1}{20}+\frac{1}{30}} = \dfrac{\frac{3}{12}}{\frac{5}{60}} = \dfrac{\frac{1}{4}}{\frac{1}{12}} = 3$ **(E)**

5. 44% of 75 plus 38% of 50 is equal to 75% of 44 plus 50% of 38, which is 33 + 19 = 52. **(B)**

6. $78\ 428 = 7 \times 11\ 204$
   $91\ 770 = 7 \times 13\ 110$
   $14\ 784 = 7 \times 2\ 112$
   $21\ 764 = 7 \times 3\ 109 + 1$
   $35\ 504 = 7 \times 5\ 072$

   So only 21 764 is not divisible by 7. **(D)**

7. $\sqrt[4]{0.0256} = (10^{-4} \times 256)^{\frac{1}{4}} = (10^{-4} \times 2^8)^{\frac{1}{4}} = 10^{-1} \times 2^2 = 0.4$ **(D)**

8. The powers 8, 28 and 12 are all multiples of 4:
   $11^8 = (11^2)^4 = 121^4,\ 5^{12} = (5^3)^4 = 125^4$ and $2^{28} = (2^7)^4 = 128^4$.
   Since $121 < 125 < 128$, i.e. $11^2 < 5^3 < 2^7$, it follows that
   $11^8 < 5^{12} < 2^{28}$. **(C)**

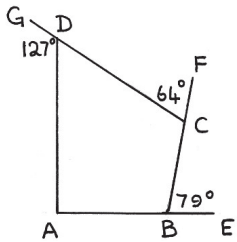
9. Since $451 = 11 \times 44,\ 452 = 2 \times 226,\ 459 = 9 \times 51$ and $465 = 5 \times 93$, the remaining number 467 has to be prime. **(E)**

10. $\dfrac{2}{\sqrt{2}} + \dfrac{1}{1+\sqrt{2}} + (1-\sqrt{2})^2 = \sqrt{2} + (\sqrt{2}-1) + (3-2\sqrt{2}) = 2$
    Alternatively:
    $\dfrac{2}{\sqrt{2}} + \dfrac{1}{\sqrt{2}+1} + (1-\sqrt{2})^2 = \sqrt{2} + \dfrac{1}{\sqrt{2}+1} + 3 - 2\sqrt{2}$
    $= \dfrac{1}{\sqrt{2}+1} + 3 - \sqrt{2} = \dfrac{1+(\sqrt{2}+1)(3-\sqrt{2})}{\sqrt{2}+1} = \dfrac{2+2\sqrt{2}}{1+\sqrt{2}} = 2$ **(D)**

11. The sum of all 80 numbers is $80 \times 79 = 6320$.
    The sum of the first 55 numbers is $55 \times 74 = 4070$.
    So the sum of the remaining 25 numbers is $6320 - 4070 = 2250$, and their average is $2250 \div 25 = 90$. **(D)**

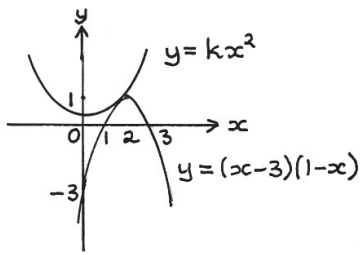
SOLUTIONS 5

1. Since $\angle ABF = 101°$, $\angle BCD = 116°$ and $\angle CDA = 53°$, $\angle BAD = 360° - 101° - 116° - 53° = 90°$.



Alternatively: Produce DA to H. With $\angle BAD = x$, then $(180° - x) + 79° + 64° + 127° = 360°$ (external angles of quadrilateral), and so $x = 180° + 79° + 64° + 127° - 360° = 90°$. **(D)**

2. When the two parabolas meet, the equation $kx^2 = (x-3)(1-x)$ is satisfied. This equation simplifies to $(k+1)x^2 - 4x + 3 = 0$.
When the parabolas are tangent to each other, the equation has just one root, so its discriminant is zero. From $16 - 4 \cdot 3(k+1) = 0$ it follows that $k = \frac{1}{3}$.



(B)

3. At the outset it must be noted that, for the log functions to be defined, we require $x > 3$ in what follows. With that in mind:

$\log_2 x + \log_2(x-3) < 2$ and $x > 3$

$\Leftrightarrow \log_2 x(x-3) < 2$ and $x > 3$

$\Leftrightarrow x(x-3) < 2^2$ and $x > 3$

$\Leftrightarrow x^2 - 3x - 4 < 0$ and $x > 3$

$\Leftrightarrow (x-4)(x+1) < 0$ and $x > 3$

$\Leftrightarrow -1 < x < 4$ and $x > 3$

$\Leftrightarrow 3 < x < 4$. **(D)**

4. One way of tackling the problem is to solve the equation, finding that the roots are $1 \pm 2\sqrt{2}$. So the new equation has roots $2 \pm 4\sqrt{2}$, and will therefore be $(x - 2 - 2\sqrt{2})(x - 2 + 2\sqrt{2}) = 0$, which when multiplied out gives $x^2 - 4x - 4 = 0$. This involves a lot of work.

A better method is to note that if an equation of the form $f(x) = 0$ is satisfied by a number a, then the equation $f(x-1) = 0$ is satisfied