

GRADE

10-12

IEB

3-in-1

Advanced Programme Mathematics IEB

Book 2: Solutions

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THIS CLASS TEXT & STUDY GUIDE INCLUDES

Full Solutions to Exercises and Exam questions:

- Statistics
- Finance & Modelling
- Matrices & Graph Theory

E-book
available 



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NB

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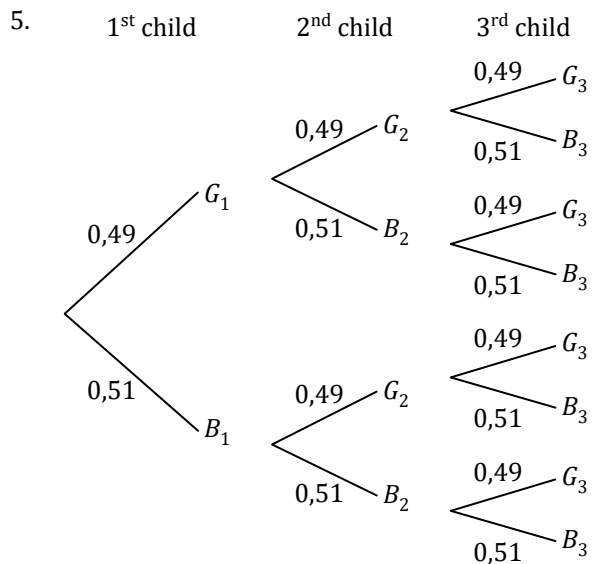
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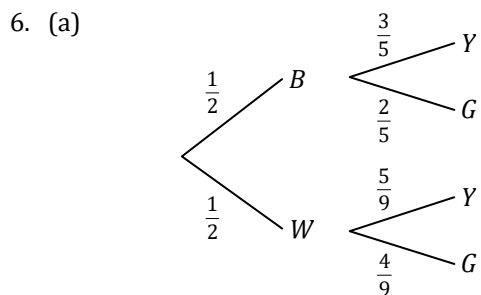
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(a) $P(G \geq 1) = 1 - P(G = 0)$
 $= 1 - 0,51 \times 0,51 \times 0,51$
 $= \mathbf{0,867349}$

(b) $1 - P(GGG) - P(BBB)$
 $= 1 - 0,49^3 - 0,51^3$
 $= \mathbf{0,74947}$

(c) $P(G_3/G_1) = P(G_3) = \mathbf{0,49}$



(b) $P(B \cap Y) = \frac{1}{2} \times \frac{3}{5} = \frac{\mathbf{3}}{\mathbf{10}}$

(c) $P(\overline{B \cap G}) + P(W \cap G) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{9}$
 $= \frac{\mathbf{19}}{\mathbf{45}}$

(d) $P(Y/W) = \frac{\mathbf{5}}{\mathbf{9}}$

(e) $P(B/Y) = \frac{P(B \cap Y)}{P(Y)}$
 $= \frac{\frac{1}{2} \times \frac{3}{5}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}}$
 $= \frac{\mathbf{27}}{\mathbf{52}}$

COUNTING METHODS

Exercise STAT 1.5

(Questions on Stat. 14)

1. (a) $\frac{6!}{2! 2!} = \mathbf{180}$

(b) $\frac{7!}{3!} = \mathbf{840}$

(c) $\frac{8!}{3! 2!} = \mathbf{3\ 360}$

(d) $\frac{13!}{3! 3!} = \mathbf{172\ 972\ 800}$

2. $8! 2! = \mathbf{80\ 640}$

3. $\frac{10!}{4! 3!} = \mathbf{25\ 200}$

4. $\frac{9!}{2! 2!} = \mathbf{90\ 720}$

5. (a) $8! = \mathbf{40\ 320}$

(b) $5! \times 2 \times 2 \times 2 = \mathbf{960}$

6. $7! = \mathbf{5\ 040}$

7. $10! 3! 2! = \mathbf{43\ 545\ 600}$

8. $4! = \mathbf{24}$

9. (a) $4! \times 2 \times 2 \times 2 \times 2 = \mathbf{384}$

(b) $4! 4! 2! = \mathbf{1\ 152}$

10. $6! = \mathbf{720}$

Exercise STAT 1.6

(Questions on Stat. 16)

1. (a) $6^4 = \mathbf{1\ 296}$

(b) $6P4 = \mathbf{360}$

2. (a) $26^3 \times 10^2 = \mathbf{1\ 757\ 600}$

(b) $26P3 \times 10P2 = \mathbf{1\ 404\ 000}$

3. $25P3 = \mathbf{13\ 800}$

4. $8P4 = \mathbf{1\ 680}$

5. $75P25 \approx \mathbf{8,15712 \times 10^{44}}$

6. $10P2 = \mathbf{90}$

7. (a) $10^5 = \mathbf{100\ 000}$

(b) $10P5 = \mathbf{30\ 240}$

8. $5P4 = \mathbf{120}$

9. $5^3 \times 3 = \mathbf{375}$

10. $10P3 = \mathbf{720}$



SIMPLE PROBABILITY DENSITY FUNCTIONS

Exercise STAT 2.7

(Questions on Stat. 48)

$$1. (a) f(x) = \begin{cases} \frac{1}{10}; & 3 < x < 13 \\ 0; & \text{elsewhere} \end{cases}$$

$$(b) P(5,2 < X < 7,1) = (7,1 - 5,2) \times \frac{1}{10} = \mathbf{0,19}$$

$$(c) P(X > 8,7) = (13 - 8,7) \times \frac{1}{10} = \mathbf{0,43}$$

$$(d) \frac{m-3}{10} = \frac{1}{2} \\ \therefore m = \mathbf{8}$$

$$2. (a) \frac{1}{k}(10 - 2) = 1 \\ \therefore \frac{8}{k} = 1 \\ \therefore k = \mathbf{8}$$

$$(b) P(X < 3,5) = \frac{1}{8}(3,5 - 2) \\ = \frac{\mathbf{3}}{16}$$

$$(c) P(X > 7,5) = \frac{1}{8}(10 - 7,5) \\ = \frac{\mathbf{5}}{16}$$

$$(d) P(5,5 < X < 6,5) = \frac{1}{8}(6,5 - 5,5) \\ = \frac{\mathbf{1}}{8}$$

$$(e) \frac{1}{8}(m - 2) = \frac{1}{2} \\ \therefore m - 2 = 4 \\ \therefore m = \mathbf{6}$$

$$3. (a) \frac{1}{7}(10 - k) = 1 \\ \therefore 10 - k = 7 \\ \therefore k = \mathbf{3}$$

$$(b) P(X > 9,3) = \frac{1}{7}(10 - 9,3) \\ = \frac{\mathbf{1}}{10}$$

$$(c) P(X < 5,2) = \frac{1}{7}(5,2 - 3) \\ = \frac{\mathbf{11}}{35}$$

$$(d) P(6,4 < X < 7,1) = \frac{7,1 - 6,4}{7} \\ = \frac{\mathbf{1}}{10}$$

$$(e) \frac{m-3}{7} = \frac{1}{2} \\ \therefore m = \frac{\mathbf{13}}{2}$$

$$4. (a) \frac{k-5}{8} = 1 \\ \therefore k - 5 = 8 \\ \therefore k = \mathbf{13}$$

$$(b) P(X > 7,1) = \frac{13 - 7,1}{8} \\ = \frac{\mathbf{59}}{80}$$

$$(c) P(X < 6,7) = \frac{6,7 - 5}{8} \\ = \frac{\mathbf{17}}{80}$$

$$(d) P(7,9 < X < 8,5) = \frac{8,5 - 7,9}{8} \\ = \frac{\mathbf{3}}{40}$$

$$(e) P(X < 4) = \mathbf{0}$$

$$(f) \frac{m-5}{8} = \frac{1}{2} \\ \therefore m = \mathbf{9}$$

Exercise STAT 2.8

(Questions on Stat. 50)

1. (a) Function is positive in given interval.

$$\frac{(4-2)}{2} = 1 \quad \therefore \text{is a p.d.f.}$$

(b) Function is positive in given interval.

$$\frac{(3-1)}{2} \cdot \left(\frac{1}{4} + \frac{3}{4}\right) = 1 \quad \therefore \text{is a p.d.f.}$$

(c) Function is positive in given interval.

$$\frac{(3-0)}{2} \cdot \left(\frac{0}{6} + \frac{1}{12} + \frac{3}{6} + \frac{1}{12}\right) = 1 \quad \therefore \text{is a p.d.f.}$$

$$(d) f(x) = \frac{1}{2}(2x - 1) \quad 0 < x < 2 \\ = x - \frac{1}{2}$$

Function is negative for $x \in \left(0; \frac{1}{2}\right)$. \therefore not a p.d.f.

NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

Exercise STAT 3.4

(Questions on Stat. 80)

1. (a) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= \binom{50}{0} 0,2^0 0,8^{50} + \binom{50}{1} 0,2^1 0,8^{49} + \binom{50}{2} 0,2^2 0,8^{48}$$

$$= \mathbf{0,001285}$$

(b) $np = 50 \times 0,2 = 10 > 5$

$$nq = 50 \times 0,8 = 40 > 5$$

$$\mu = np = 10; \quad \sigma = \sqrt{50 \times 0,2 \times 0,8} \approx 2,8284$$

$$P(X \leq 2) \rightarrow P(X < 2,5)$$

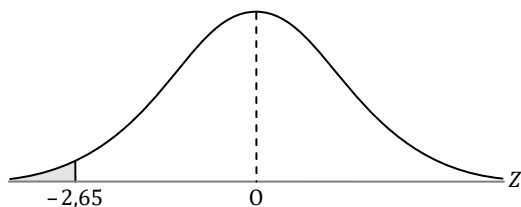
$$= P\left(Z < \frac{2,5 - 10}{2,8284}\right)$$

$$= P(Z < -2,65)$$

$$= 0,5 - H(2,65)$$

$$= 0,5 - 0,49598$$

$$= \mathbf{0,00402}$$



(c) $P(X \geq 10) \rightarrow P(X > 9,5)$

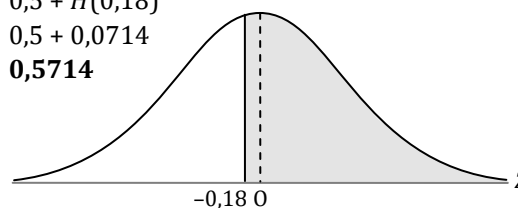
$$= P\left(Z > \frac{9,5 - 10}{2,8284}\right)$$

$$= P(Z > -0,18)$$

$$= 0,5 + H(0,18)$$

$$= 0,5 + 0,0714$$

$$= \mathbf{0,5714}$$



(d) $P(X > 7) \rightarrow P(X > 7,5)$

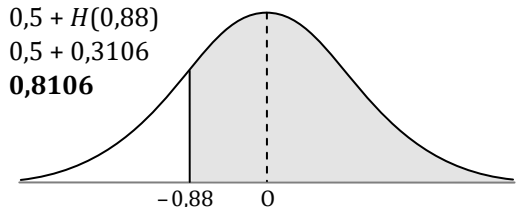
$$= P\left(Z > \frac{7,5 - 10}{2,8284}\right)$$

$$= P(Z > -0,88)$$

$$= 0,5 + H(0,88)$$

$$= 0,5 + 0,3106$$

$$= \mathbf{0,8106}$$



2. (a) $np = 0,72 \times 50 = 36 > 5$

$$nq = 0,28 \times 50 = 14 > 5$$

$$\mu = 36; \quad \sigma \approx 3,1749$$

\therefore A normal approximation may be used.

$$P(X \geq 30) \rightarrow P(X > 29,5)$$

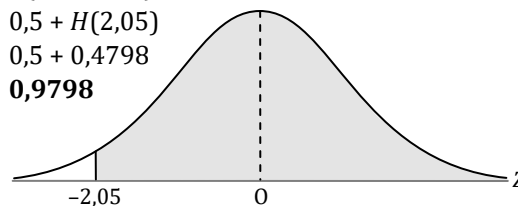
$$= P\left(Z > \frac{29,5 - 36}{3,1749}\right)$$

$$= P(Z > -2,05)$$

$$= 0,5 + H(2,05)$$

$$= 0,5 + 0,4798$$

$$= \mathbf{0,9798}$$



(b) $P(X > 25) \rightarrow P(X > 25,5)$

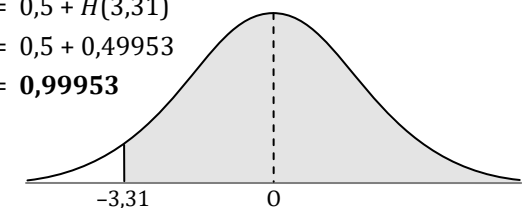
$$= P\left(Z > \frac{25,5 - 36}{3,1749}\right)$$

$$= P(Z > -3,31)$$

$$= 0,5 + H(3,31)$$

$$= 0,5 + 0,49953$$

$$= \mathbf{0,99953}$$



(c) $P(X \leq 35) \rightarrow P(X < 35,5)$

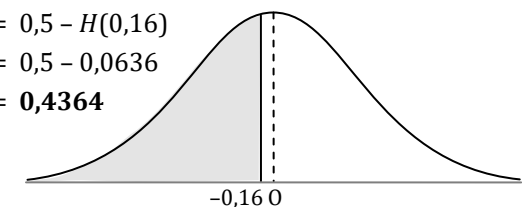
$$= P\left(Z < \frac{35,5 - 36}{3,1749}\right)$$

$$= P(Z < -0,16)$$

$$= 0,5 - H(0,16)$$

$$= 0,5 - 0,0636$$

$$= \mathbf{0,4364}$$



3. $np = 400 \times 0,015 = 6 > 5$

$$nq = 400 \times 0,985 = 394 > 5$$

$$\therefore \mu = 6; \quad \sigma = \sqrt{400 \times 0,015 \times 0,985} \approx 2,431$$

$$P(X > 5) \rightarrow P(X > 5,5)$$

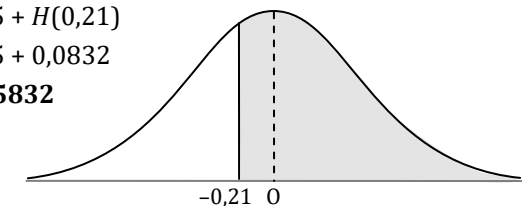
$$= P\left(Z > \frac{5,5 - 6}{2,431}\right)$$

$$= P(Z > -0,21)$$

$$= 0,5 + H(0,21)$$

$$= 0,5 + 0,0832$$

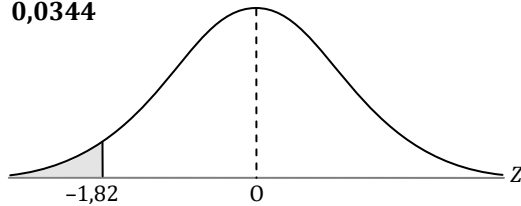
$$= \mathbf{0,5832}$$



Chapter 3: Exercise STAT 3.4 & 3.5

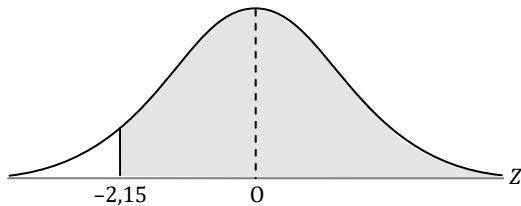
4. $np = 325 \times 0,11 = 35,75 > 5$
 $nq = 325 \times 0,89 = 289,25 > 5$
 $\therefore \mu = 35,75; \quad \sigma \approx 5,6407$

$P(X \leq 25) \rightarrow P(X < 25,5)$
 $= P\left(Z < \frac{25,5 - 35,75}{5,6407}\right)$
 $= P(Z < -1,82)$
 $= 0,5 - H(1,82)$
 $= 0,5 - 0,4656$
 $= \mathbf{0,0344}$



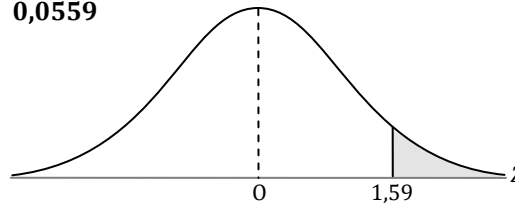
5. $np = 150 \times 0,66 = 99 > 5$
 $nq = 150 \times 0,34 = 51 > 5$
 $\therefore \mu = 99; \quad \sigma \approx 5,8017$

$P(X \geq 87) \rightarrow P(X > 86,5)$
 $= P\left(Z > \frac{86,5 - 99}{5,8017}\right)$
 $= P(Z > -2,15)$
 $= 0,5 + H(2,15)$
 $= 0,5 + 0,4842$
 $= \mathbf{0,9842}$

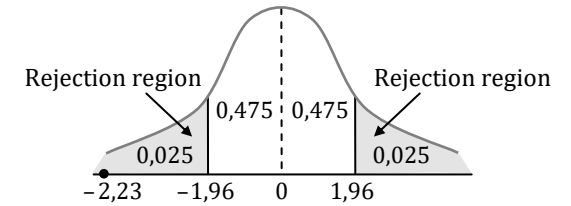


6. $np = 50 \times 0,4 = 20 > 5$
 $nq = 50 \times 0,6 = 30 > 5$
 $\therefore \mu = 20; \quad \sigma \approx 3,4641$

$P(X > 25) \rightarrow P(X > 25,5)$
 $= P\left(Z > \frac{25,5 - 20}{3,4641}\right)$
 $= P(Z > 1,59)$
 $= 0,5 - H(1,59)$
 $= 0,5 - 0,4441$
 $= \mathbf{0,0559}$



(b) $H_0: \mu = 51 \quad H_1: \mu \neq 51$
 $\therefore X \sim N(51; 5,5^2)$



$z = \frac{50 - 51}{\frac{5,5}{\sqrt{150}}} \approx -2,23$

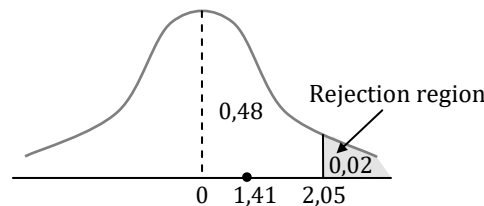
Lies in rejection region $(-2,23 < -1,96)$.

\therefore **Reject H_0 at a 5% significance level.**
Sufficient evidence to support the claim.

HYPOTHESIS TESTING

Exercise STAT 3.5 (Questions on Stat.85)

1. (a) $H_0: \mu = 14 \quad H_1: \mu > 14$
 $\therefore X \sim N(14; 9)$

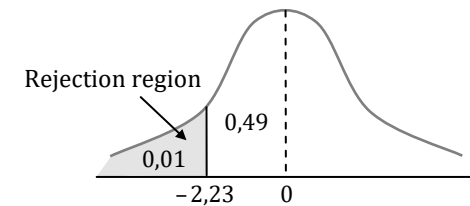


$z = \frac{14,3 - 14}{\frac{3}{\sqrt{200}}} \approx 1,41$

Does not lie in rejection region $(1,41 < 2,05)$.

\therefore **Do not reject H_0 at a 2% significance level.**
Insufficient evidence to support the claim.

(c) $H_0: \mu = 1,75 \quad H_1: \mu < 1,75$
 $\therefore X \sim N(1,75; 0,5^2)$

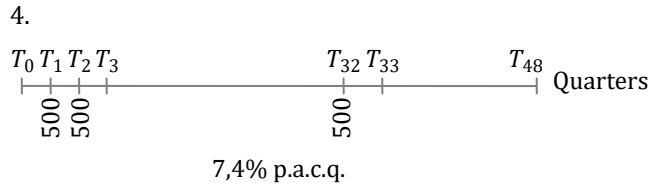


$z = \frac{1,5 - 1,75}{\frac{0,5}{\sqrt{49}}} = -3,5$

Lies in rejection region $(-3,5 < -2,23)$.

\therefore **Reject H_0 at a 1% significance level.**
Sufficient evidence to support the claim.

Chapter 2: Exercise FIN 2.3 & 2.4



For the first 8 years we use the formula

$$F = x \frac{(1+i)^n - 1}{i}$$

After that, for the last 4 years we use the formula

$$F = P(1+i)^n,$$

$$1 + \frac{0,074}{4} = \frac{2\,037}{2\,000} \rightarrow \text{MEM A}$$

$$F = 500 \frac{A^{32-1}}{A-1} \cdot A^{16} \approx \mathbf{R28\,913,14}$$

5. $1 + \frac{0,052}{12} = \frac{3\,013}{3\,000} \rightarrow \text{MEM A}$

$$205\,772,30 = \frac{A^{24 \times 12} - 1}{A - 1} \cdot A^{6 \times 12}$$

$$= x(779,4\,405\dots)$$

$$\therefore x \approx \mathbf{R264 \text{ per month}}$$

6. (a) Depreciated Value = $420\,000(1 - 0,11)^6$
 $\approx \mathbf{R208\,732,14} \rightarrow \text{MEM B}$

(b) Cost of new van = $420\,000(1 + 0,055)^6$
 $\approx \mathbf{R579\,113,98} \rightarrow \text{MEM C}$

(c) $1 + \frac{0,073}{12} = \frac{12\,073}{12\,000} \rightarrow \text{MEM A}$

$$\text{Shortfall} = 370\,381,84$$

$$= x \frac{[A^{6 \times 12} - 1]}{A - 1}$$

$$= x(90,008\dots)$$

$$\therefore x \approx \mathbf{R4\,114,98 \text{ per month}}$$

7. (a) Depreciated value = $12\,000\,000(1 - 0,104)^5$
 $\approx \mathbf{R6\,929\,809,47}$

(b) Replacement Value = $12\,000\,000(1 + 0,062)^5$
 $\approx \mathbf{R16\,210\,776,93}$

(c) Shortfall = $16\,210\,776,93 - 6\,929\,809,47$
 $= \mathbf{R9\,280\,967,47}$

$$1 + \frac{0,046}{4} = \frac{2\,023}{2\,000} \rightarrow \text{MEM A}$$

$$9\,280\,967,47 = x \cdot \frac{[A^{12} - 1]}{A - 1} (A)^8$$

$$= x(14,013\,899)$$

$$\therefore x \approx \mathbf{R662\,268,74 \text{ per quarter}}$$

Exercise FIN 2.4

(Questions on Fin.26)

1. $1 + \frac{0,14}{12} = \frac{607}{600} \rightarrow \text{MEM A}$

At T_5 : $40\,000A^5 = \frac{x[1 - A^{-19}]}{A - 1}$

$$\therefore 42\,388,41\dots = x(16,953\dots)$$

$$\therefore x \approx \mathbf{R2\,500,30 \text{ per month}}$$

2. $1 + \frac{0,08}{12} = \frac{151}{150} \rightarrow \text{MEM A}$

At T_{23} : $100\,000A^{23} = \frac{x[1 - A^{-37}]}{A - 1}$

$$\therefore 116\,512,0462 = x(32,693\dots)$$

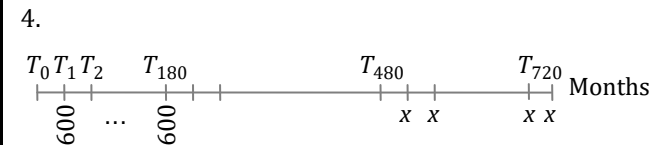
$$\therefore x \approx \mathbf{R3\,563,73 \text{ per month}}$$

3. $1 + \frac{0,068}{12} = \frac{3\,017}{3\,000} \rightarrow \text{MEM A}$

At T_3 : $P \cdot A^3 = 680 \cdot \frac{[1 - A^{-57}]}{A - 1}$

$$\therefore P(1,017\,096\dots) = 33\,043,89\,364$$

$$\therefore P \approx \mathbf{R32\,488,45}$$



$$1 + \frac{0,076}{12} = \frac{3\,019}{3\,000} \rightarrow \text{MEM A}$$

At T_{480} : $\frac{600[A^{180} - 1]}{A - 1} \cdot A^{300} = \frac{x[1 - A^{-240}]}{A - 1}$

$$\therefore 1\,331\,985,047 = x(123,195\dots)$$

$$\therefore x \approx \mathbf{R10\,811,97}$$

5. $1 + \frac{0,5}{12} = \frac{25}{24} \rightarrow \text{MEM A}$

At T_{18} :

Bob's O.B.:

$$50\,000A^{18} - \frac{4\,000[A^{18} - 1]}{A - 1} = 87,4\,059\dots \rightarrow \text{MEM B}$$

Abie's O.B.:

$$50\,000A^{18} - \frac{4\,000[A^{12} - 1]}{A - 1} = 43\,571,78\,285 \rightarrow \text{MEM C}$$

$$\text{Difference} \approx \mathbf{R43\,484,38}$$



$$(b) 1 + \frac{0,105}{12} = \frac{807}{800} \rightarrow \text{MEM C}$$

At T_9 :

$$\begin{aligned} \text{O.B.} &= \frac{10\,000[1 - A^{-231}]}{A - 1} \\ &= 1\,023\,547,306 \rightarrow \text{MEM D} \end{aligned}$$

$$D = \frac{x[1 - C^{-231}]}{C - 1} \quad \text{SOLVE: Appendix 1}$$

$$= x(99,0103\dots)$$

$$\therefore x = \text{R}10\,337,78 \text{ per month}$$

$$\therefore \text{Increase} = \text{R}337,78$$

$$(c) 1 + \frac{0,115}{12} = \frac{2\,423}{2\,400} \rightarrow \text{MEM E}$$

At T_{24} :

$$\begin{aligned} \text{O.B.} &= \frac{10\,337,78[1 - C^{-216}]}{C - 1} \\ &= 1\,001\,501,84 \rightarrow \text{MEM F} \end{aligned}$$

$$F = \frac{10\,337,78[1 - E^{-n}]}{E - 1} \quad \text{SOLVE: Appendix 1}$$

$$\therefore 1 - E^{-n} = 0,92841\dots$$

$$\therefore E^{-n} = 0,07158\dots$$

$$\therefore -n = \log_E[\text{ANS}]$$

$$= -276,46\dots$$

$$\text{Total no. of payments} = 24 + 277 = 301$$

$$\text{Extra months: } 301 - 240 = 61$$

i.e. **5 years 1 month**

(d) Interest on O.B. for 1 month

$$= F \left(\frac{0,125}{12} \right) = 10\,432,31\dots > 10\,337,78$$

Her monthly payments will not cover the interest charges, resulting in her outstanding balance becoming larger each month. She will never be able to pay off the loan, thus the bank cannot grant an extension.

$$3. 1 + \frac{0,115}{12} = \frac{2\,423}{2\,400} \rightarrow \text{MEM A}$$

At T_0 :

$$50\,000 = \frac{1\,000[1 - A^{-12}]}{A - 1} \quad \text{SOLVE: Appendix 1}$$

$$+ \frac{1\,200[1 - A^{-12}]}{A - 1} \cdot A^{-12}$$

$$+ \frac{1\,500[1 - A^{-n}]}{A - 1} \cdot A^{-24}$$

$$\therefore \frac{1\,500[1 - A^{-n}]}{A - 1} \cdot A^{-24} = 26\,637,99637$$

$$\therefore 1 - A^{-n} = 0,2139\dots$$

$$\therefore A^{-n} = 0,786\dots$$

$$\therefore -n = \log_A[\text{ANS}]$$

$$= -25,242\dots$$

$$\therefore n = 25,242\dots$$

$$\text{Total no. of payments} = 24 + 26 = \mathbf{40}$$

$$4. (a) 1 + \frac{0,09}{12} = \frac{403}{400} \rightarrow \text{MEM A}$$

At T_{120} :

$$F = \frac{200[A^{48} - 1]}{A - 1} \cdot A^{72} + \frac{500[A^{72} - 1]}{A - 1}$$

$$\approx \text{R}67\,204,96$$

(b) At T_{n+48} :

$$100\,000 = \frac{200[A^{48} - 1]}{A - 1} \cdot A^n + \frac{500[A^n - 1]}{A - 1}$$

SOLVE: Appendix 1

$$\therefore 100\,000(A - 1) = (86,281\dots)A^n + 500A^n - 500$$

$$\therefore (586,628\,1\dots)A^n = 1\,250 \quad \text{MEM B}$$

$$\therefore A^n = 2,132\dots$$

$$\therefore n = \log_A[\text{ANS}] = 101,324\dots$$

$$\text{Total no. of payments} = 48 + 102 = 150$$

i.e. It will take **150 months**

= **12 years 6 months**

Exercise FIN 3.4

(Questions on Fin. 42)

1. Fig. 1 - (b)

Fig. 2 - (d)

Fig. 3 - (a)

Fig. 4 - (c)

$$2. 1 + \frac{0,075}{12} = \frac{161}{160} \rightarrow \text{MEM A}$$

$$1 + i_{\text{eff}} = A^{12} = 1,077\dots \rightarrow \text{MEM B}$$

$$\text{At } T_{120}: F = \frac{500[A^{120} - 1]}{A - 1} + \frac{500[B^{10} - 1]}{B - 1}$$

$$\approx \text{R}96\,127,53$$

$$3. 1 + \frac{0,21}{12} = \frac{407}{400} \rightarrow \text{MEM A}$$

$$9\,500 = 1\,000A^{-24} + \frac{300[1 - A^{-n}]}{A - 1}$$

$$\therefore 1 - A^{-n} = 0,515\dots$$

$$\therefore A^{-n} = 0,4843\dots$$

$$\therefore -n = \log_A[\text{ANS}]$$

$$= -41,7928\dots$$

\therefore **42 months**

MODELLING SOLUTIONS

CHAPTER 4: GR 10 MODELLING

Exercise MOD 4.1

(Questions on Mod.4)

1. (a) $T_{34} = 87$ (b) $T_2 = T_1 + 8$
 (c) $T_{13} = 2 \cdot T_{12} - 3$ (d) $T_n = 3 \cdot T_{n-1} + 1$

2. (a) 75, 70, 65, 60, 55
 $T_n = T_{n-1} - 5$, $T_1 = 75$

(b) 11, 25, 53, 109, 221
 $T_n = 2 \cdot T_{n-1} + 3$, $T_1 = 11$

(c) 144, 84, 54, 39, 31,5
 $T_n = \frac{1}{2} \cdot T_{n-1} + 12$, $T_1 = 144$

(d) -1, -3, -7, -15, -31
 $T_n = 2 \cdot T_{n-1} - 1$, $T_1 = -1$

3. (a) -6, 2, 10, 18, 24
 (b) 3, -11, 31, -95, 283
 (c) -8, 2, 7, 9,5, 10,75
 (d) 3, 2, 0, -4, -12
 (e) 24, -8, 0, -2, -1,5
 (f) 0, 5, 11, 18, 26



4. (a) $T_n = 3n - 1$ $T_n = T_{n-1} + 3$, $T_1 = 2$
 $T_{17} = 50$

(b) $T_n = 4^n$ $T_n = 4 \cdot T_{n-1}$, $T_1 = 4$
 $T_{17} = 17\ 179\ 869\ 184$

(c) $T_n = (-3)^{n-1}$ $T_n = -3 \cdot T_{n-1}$, $T_1 = 1$
 $T_{17} = 43\ 046\ 721$

(d) $T_n = 2,5 \cdot n - 6,5$ $T_n = T_{n-1} + 2,5$, $T_1 = -4$
 $T_{17} = 36$

(e) $T_n = 5n - 4$ $T_n = T_{n-1} + 5$, $T_1 = 1$
 $T_{17} = 81$

(f) $T_n = 5^{n-3}$ $T_n = 5 \cdot T_{n-1}$, $T_1 = 0,04$
 $T_{17} = 6\ 103\ 515\ 625$

5. (a) $28 = 10k + c$ ①
 $10 = 4k + c$ ②
 $18 = 6k$ ① - ②
 $\therefore k = 3$, $c = -2$
 $T_n = 3 \cdot T_{n-1} - 2$, $T_1 = 2$ $T_{12} = 177\ 148$

(b) $17 = 9k + c$ ①
 $9 = 5k + c$ ②
 $8 = 4k$ ① - ②
 $\therefore k = 2$, $c = -1$

$T_n = 2 \cdot T_{n-1} - 1$, $T_1 = 3$ $T_{12} = 4\ 097$

(c) $6 = 20k + c$ ①
 $20 = 48k + c$ ②
 $14 = 28k$ ② - ①
 $\therefore k = \frac{1}{2}$, $c = -4$

$T_n = \frac{1}{2} \cdot T_{n-1} - 4$, $T_1 = 48$
 $T_{12} \approx -7,973$

(d) $-228 = 60k + c$ ①
 $60 = -12k + c$ ②
 $288 = -72k$ ② - ①
 $\therefore k = -4$, $c = 12$

$T_n = -4 \cdot T_{n-1} + 12$, $T_1 = 6$
 $T_{12} = -15\ 099\ 492$

(e) $17 = 33k + c$ ①
 $33 = 81k + c$ ②
 $16 = 48k$ ② - ①

$\therefore k = \frac{1}{3}$, $c = 6$

$T_n = \frac{1}{3} \cdot T_{n-1} + 6$, $T_1 = 225$ $T_{12} \approx 9,001$

(f) $72 = -16k + c$ ①
 $-16 = 6k + c$ ②
 $88 = -22k$ ① - ②
 $\therefore k = 4$, $c = 8$

$T_n = -4 \cdot T_{n-1} + 8$, $T_1 = \frac{1}{2}$ $T_{12} = 4\ 613\ 736$

6. (a) (i) 6, 10, 14, 18, 22, 26
 (ii) $T_n = T_{n-1} + 4$, $T_1 = 6$
 (iii) $T_n = 4n + 2$

(b) (i) 3, 5, 7, 9, 11, 13
 (ii) $T_n = T_{n-1} + 2$, $T_1 = 3$
 (iii) $T_n = 2n + 1$

(c) (i) 5, 8, 11, 14, 17, 20
 (ii) $T_n = T_{n-1} + 3$, $T_1 = 5$
 (iii) $T_n = 3n + 2$

7. (a) $T_n = 1,2 \cdot T_{n-1}$, $T_1 = 50$

(b) $T_7 = 149,2992$ mm

8. (a) 2, 4, 8, ... $T_n = 2^n$

(b) $T_{12} = 2^{12} = 4\ 096$

(c) $T_n = 2 \cdot T_{n-1}$, $T_1 = 2$

(d) $T_{10} = 1\ 024$ $n = 10$



CHAPTER 6: GR 12 MODELLING

Exercise MOD 6.1

(Questions on Mod.43)

1.1 annual cycle

$$K = 25\ 000$$

$$c = \frac{1}{20} = 0,05$$

$$a = 0,5 \times 2 \times 0,6 \quad a = 0,6$$

$$b \cdot P_0 \cdot S_0 = 4\ 500 = b(6\ 000)(1\ 500) \quad b = 0,000\ 5$$

$$f \cdot b \cdot P_0 \cdot S_0 = 100 = f(0,000\ 5)(6\ 000)(1\ 500) \quad f = 0,022$$

1.2 annual cycle

$$K = 2 \times 500 = 1\ 000$$

$$c = \frac{1}{5} = 0,2$$

$$a = 0,65 \times (7 \times 1) \times 0,4 \quad a = 1,82$$

$$b \cdot B_0 \cdot S_0 = 36 \times 1 \times 52 = 1\ 872$$

$$1\ 872 = b(500)(36) \quad b = 0,104$$

$$f \cdot b \cdot B_0 \cdot S_0 = (0,5 \times 36) \times 6 = 108$$

$$108 = f(0,104)(500)(36) \quad f = 0,058$$

1.3 annual cycle

$$K = 45\ 000$$

$$c = \frac{1}{30} = 0,033$$

$$a = 0,65 \times 1 \times 0,7 \quad a = 0,455$$

$$b \cdot S_0 \cdot W_0 = 30 \times 1 \times 52 = 1\ 560$$

$$1\ 560 = b(30\ 000)(30) \quad b = 0,001\ 733$$

$$f \cdot b \cdot S_0 \cdot W_0 = (0,6 \times 30) \times \left(\frac{4}{3}\right) = 24$$

$$24 = f(0,001\ 733)(30\ 000)(30) \quad f = 0,015\ 388$$

1.4 annual cycle

$$K = 180\ 000$$

$$c = \frac{1}{12} = 0,083$$

$$a = 0,5 \times 1 \times 0,7 \quad a = 0,35$$

$$b \cdot I_0 \cdot D_0 = 350 \times 2 \times 12 = 8\ 400$$

$$8\ 400 = b(100\ 000)(350) \quad b = 0,000\ 24$$

$$f \cdot b \cdot I_0 \cdot D_0 = (0,5 \times 0,1 \times 350 \times 10) = 70$$

$$70 = f(0,000\ 24)(100\ 000)(350) \quad f = 0,083$$

1.5 two-year cycle

$$K = 2 \times 500 = 1\ 000$$

$$c = \frac{2}{10} = 0,2$$

$$a = 0,3 \times (3 \times 3) \times 0,45 \quad a = 1,215$$

$$b \cdot D_0 \cdot E_0 = 30 \times 1 \times 24 = 720$$

$$720 = b(500)(30) \quad b = 0,048$$

$$f \cdot b \cdot D_0 \cdot E_0 = 20 = f(0,048)(500)(30)$$

$$f = 0,028$$

2.1 28 < mouse < 70

2.2 equilibrium: amplitude of each cycle diminishes

2.3 about 35 months

2.4 no: initial drop is not that dramatic

3.1 amplitude of each cycle diminishes

3.2 160 to 200 months

3.3 about 35 months

3.4 deer (prey) population initially quite high (200) when compared to what it is tapering off to (60).

OR

Large amount of deer roaming around makes them ready prey for wolves.

3.5 18 < predator < 48

4.1 badgers \approx 12, squirrels \approx 400

4.2 8 < badgers < 20

4.3 about 370

4.4 rapid initial drop (500 to 200) in squirrel population probably caused by great number of badgers, which remains constant for quite a while.

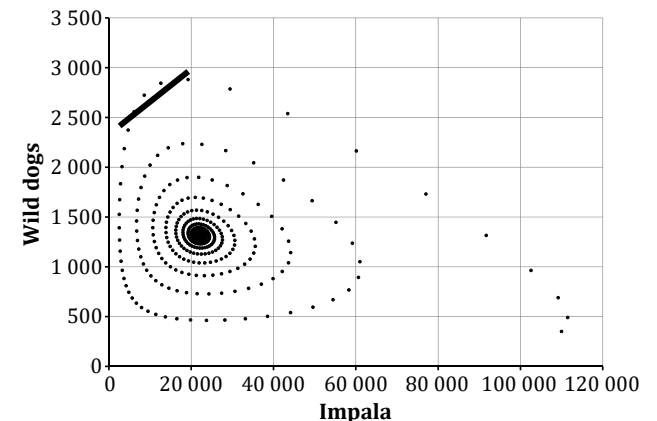
4.5 For equilibrium population for prey: $S_n = \frac{c}{f \cdot b}$

$c = \frac{1}{\text{lifespan}}$ if lifespan increases, c decreases and

hence S_n decreases.

5.1 $I_0 = 110\ 000$ and $D_0 = 350$

5.2



CHAPTER 2: Gr 11 MATRICES

Exercise MAT 2.1 (Identifying Transformations)

(Questions on Mat. 21)

1. A rotation of 270° $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

B reflection across $y = x$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

C shear with y -axis as invariant, by a factor of 2 $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

D reflection across $x = 0$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

E shear with x -axis as invariant, by a factor of 3 $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

F stretch with x -axis as invariant, by a factor of -3 $\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$

G rotation of 90° anticlockwise $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

H stretch with y -axis as invariant, by a factor of $-2,5$ $\begin{pmatrix} -2,5 & 0 \\ 0 & 1 \end{pmatrix}$

I enlargement by a factor of 3 $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

J enlargement by a factor of $-2,5$ $\begin{pmatrix} -2,5 & 0 \\ 0 & -2,5 \end{pmatrix}$

K stretch with y -axis as invariant, by a factor of 2 $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

L shear with y -axis as invariant, by a factor of -3 $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

M enlargement by a factor of $\frac{1}{2}$ $\begin{pmatrix} 0,5 & 0 \\ 0 & 0,5 \end{pmatrix}$

N rotation of 180° $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

O reflection across $y = 0$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

P stretch with x -axis as invariant, by a factor of 3 $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$



Q shear with x -axis as invariant, by a factor of -2 $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

R reflection across $y = -x$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

2. S translation 5 units left and 3 units up $\begin{pmatrix} -5 & -5 & -5 & -5 \\ +3 & +3 & +3 & +3 \end{pmatrix}$

T translated 3 units left and 2 units down $\begin{pmatrix} -3 & -3 & -3 & -3 \\ -2 & -2 & -2 & -2 \end{pmatrix}$

U translated 1 unit right and 6 units down $\begin{pmatrix} +1 & +1 & +1 & +1 \\ -6 & -6 & -6 & -6 \end{pmatrix}$

3. V translated 5 units right and 1 unit up $\begin{pmatrix} +5 & +5 & +5 & +5 \\ +1 & +1 & +1 & +1 \end{pmatrix}$

then enlarged by a factor of 1 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

W translated 1 unit right $\begin{pmatrix} +1 & +1 & +1 & +1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

then enlarged by a factor of 2 $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

X translated 2 units right and 3 units down $\begin{pmatrix} +2 & +2 & +2 & +2 \\ -3 & -3 & -3 & -3 \end{pmatrix}$

then enlarged by a factor of 3 $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$



Exercise MAT 2.2 (Calculating Transformations)

(Questions on Mat. 23)

1. (a) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 6 & 0 \\ -3 & 0 & 9 \end{pmatrix}$ $P'(-3; -3)$, $Q'(6; 0)$ and $R'(0; 9)$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & -6 \end{pmatrix}$ $P'(-1; 2)$, $Q'(2; 0)$ and $R'(0; -6)$

(c) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ -5 & 8 & 3 \end{pmatrix}$ $P'(-1; -5)$, $Q'(2; 8)$ and $R'(0; 3)$

Chapter 2: Exercise MAT 2.2

2. (a) $\tan \theta = \sqrt{3}$ hence $\theta = 60^\circ$

$$\begin{pmatrix} \cos 2(60^\circ) & \sin 2(60^\circ) \\ \sin 2(60^\circ) & -\cos 2(60^\circ) \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$(\sqrt{3}; 1) \rightarrow (0; 2)$$

(b) $\tan \theta = 0,5$ hence $\theta = 26,565^\circ$

$$\begin{pmatrix} \cos 2(26,565^\circ) & \sin 2(26,565^\circ) \\ \sin 2(26,565^\circ) & -\cos 2(26,565^\circ) \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$(3; -1) \rightarrow (1; 3)$$

3. (a) $\begin{pmatrix} \cos 53,13^\circ & -\sin 53,13^\circ \\ \sin 53,13^\circ & \cos 53,13^\circ \end{pmatrix} \begin{pmatrix} 10 \\ 15 \end{pmatrix} = \begin{pmatrix} -6 \\ 17 \end{pmatrix}$

$$(10; 15) \rightarrow (-6; 17)$$

(b) $\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ -2 \end{pmatrix} = \begin{pmatrix} 2,5 \\ -0,87 \end{pmatrix}$

$$(\sqrt{3}; -2) \rightarrow (2,5; -0,87)$$

4. (a) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 13 & -13 & -52 \\ 0 & -26 & 39 \end{pmatrix} = \begin{pmatrix} 13 & -13 & -52 \\ 26 & -52 & -65 \end{pmatrix}$

(b) $\begin{pmatrix} \cos 22,62^\circ & -\sin 22,62^\circ \\ \sin 22,62^\circ & \cos 22,62^\circ \end{pmatrix} \begin{pmatrix} 13 & -13 & -52 \\ 0 & -26 & 39 \end{pmatrix} = \begin{pmatrix} 12 & -2 & -63 \\ 5 & -29 & 16 \end{pmatrix}$

(c) $\tan \theta = 0,5$ so, $\theta = 26,565^\circ$

$$\begin{pmatrix} \cos 2(26,565^\circ) & \sin 2(26,565^\circ) \\ \sin 2(26,565^\circ) & -\cos 2(26,565^\circ) \end{pmatrix} \begin{pmatrix} 13 & -13 & -52 \\ 0 & -26 & 39 \end{pmatrix}$$

$$= \begin{pmatrix} 7,8 & -28,6 & 0 \\ 10,4 & 5,2 & -65 \end{pmatrix}$$

5. (a) $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

Note the order of matrix multiplication is significant.



(b) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -4 & -3 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -12 & -9 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

Note the order of matrix multiplication is significant.



7. $\tan \theta = \sqrt{3}$ hence $\theta = 60^\circ$

$$\begin{pmatrix} \cos 2(60^\circ) & \sin 2(60^\circ) \\ \sin 2(60^\circ) & -\cos 2(60^\circ) \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1,732 \\ -1,732 & -1 \end{pmatrix}$$

8. (a) $\begin{pmatrix} 1,8 & -2,4 \\ 2,4 & 1,8 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 5,4 & -15,6 \\ 7,2 & 4,2 \end{pmatrix}$

(b) $T^{-1} = \frac{1}{9} \begin{pmatrix} 1,8 & 2,4 \\ -2,4 & 1,8 \end{pmatrix}$

(c) $\frac{1}{9} \begin{pmatrix} 1,8 & 2,4 \\ -2,4 & 1,8 \end{pmatrix} \begin{pmatrix} 15 \\ -30 \end{pmatrix} = \begin{pmatrix} -5 \\ -10 \end{pmatrix}$

(d) $\begin{pmatrix} \cos 53,1^\circ & -\sin 53,1^\circ \\ \sin 53,1^\circ & \cos 53,1^\circ \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1,8 & -2,4 \\ 2,4 & 1,8 \end{pmatrix}$

9. (a) $\cos \theta = 0,6$ so $\theta = 53,13^\circ$

(b) $x = -\sin 53,13^\circ = -0,8$ $y = \sin 53,13^\circ = 0,8$ $z = 0,6$

(c) $\begin{pmatrix} 0,6 & -0,8 \\ 0,8 & 0,6 \end{pmatrix} \begin{pmatrix} -5 \\ 10 \end{pmatrix} = \begin{pmatrix} -11 \\ 2 \end{pmatrix}$

10. $\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} -10 \\ 30 \end{pmatrix} = \begin{pmatrix} -18 \\ -26 \end{pmatrix}$

$$-10\cos \alpha - 30\sin \alpha = -18 \quad \text{and} \quad 30\cos \alpha - 10\sin \alpha = -26$$

$$\therefore \cos \alpha = -3/5$$

$$\sin \alpha = 4/5$$

$$\therefore \alpha = 180^\circ - 53,13^\circ = 126,87^\circ$$

11. $\cos \alpha = -\sqrt{3}/2$ so angle lies in quad II or III

$\sin \alpha = +1/2$ so angle lies in quad I or II

$$\therefore \alpha = 180^\circ - 30^\circ = 150^\circ$$

12. $\cos 2\alpha = 0,8$ so angle lies in quad I or IV

$\sin 2\alpha = 0,6$ so angle lies in quad I or II

$$2\alpha = 36,87^\circ \quad \text{and so} \quad \alpha = 18,435^\circ$$

13. $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -0,232 \\ 3,6 \end{pmatrix}$

$$3\cos 2\alpha - 2\sin 2\alpha = 0,232 \quad \text{and} \quad 2\cos 2\alpha + 3\sin 2\alpha = 3,6$$

$$\cos 2\alpha = 0,500 \quad \text{so} \quad \alpha = 29,99^\circ$$

$$\text{OR} \quad \sin 2\alpha = 0,866 \quad \text{so} \quad \alpha = 30,03^\circ$$

difference due to irrational rounding



CHAPTER 6: Gr 12 GRAPH THEORY**Exercise GT 6.1**

(Questions on GT.25)

- A D C E G J H I F B A** (many other options)
 - DF (19), DH (22), AB (25), FI (27), IJ (29), EG (30), DC (38), CE (57), AD (65) = **312 m**
- RS (2), QP (3), PW / SV / SU / TU (4) (any order), VW (6) = **R27 000**
 - WP (4), PQ (3), WV (6), SV (4), SR (2), SU (4), UT (4) = **R27 000**
- by choosing all three edges of weight 5, he has created a circuit CBGF. Vertex A is not linked to the tree
 - CD (3), CB (4), CF / FG / BG (5) (any two of the edges), DE (7), HG (8), BA (13) = **45 units**
- DG (40) + GE (45) + EH (50) + CD (80) + BF (80) + CF (85)
380 + 225 = **605 units**
 - BF (80) + FC (85) + CD (80) + DG (40) + GE (45) + EH (50) + HGA (215) + AB (150)
= **745 units**
 - B F C E H G D A B = **700 units**
OR
B F H E G D C A B = **725 units**
OR
any other route between 515 and 745
- AD = 5, CFE = 14 $72 + 19 = 91$ hours
 - AD, AE, AB, BC, 5 + 5 = **34 hours**
 - Yes, the LB can change if the initial vertex left out is changed.
 - A D F E C B A = **43**
OR
A F D E C B A = **47** (both acceptable)
 - A D E F C B A = **40** (any route between 34 and 43 fine)



- GE (4), GD (8), GJ (9), JH (6), JK (8), FE (10), AD (12), AB (10), CF (14)
= 81 units = **R81 000**
 - EG (4), GD (8), DH (10), HJ (6), JK (8), KF (12), FC (14), CA (15), AB (10), BE (25)
= 112 units = **112 minutes**
 - is quite feasible that a much better "good" route exists
The first half of the UB (36) is substantially shorter than the second half (76)
- DE (4), AB (5)/JE (5), JG (7)/EF (7), HG (8), BD (9)/CD (9), KB (6)/KJ (6)
= 54 + 12 = **66 units**
 - AB (5), BK (6), KJ (6), JE (5), ED (4), DC (9), CKH (15), HG (8), GF (9),
FEDBA (25) = **92 units**
 - A, B, K, C, D, E, J, F, G, H, A = **74 units**
OR
A, B, C, D, E, F, G, J, K, H, A = **74 units**
OR
Many other options

Gr 12 Graph Theory Exam Solutions

(Questions on GT.28)

- Dijkstra
 - Prim
 - Fleury
 - Nearest Neighbour
- 9 edges
 - Kruskal: shortest edges in graph are being chosen
 - 6
 - JE: will complete a mini-circuit
 - DE, AB, IJ, DI **EF / JG, HG / CI, BI = 60**
- PU + UQ < PQ
 - VW
 - triangular inequality does not hold
 - W V R Q U P S T W or W V R U Q P S T W
 - PSTW = 15
 $2 + 3 + x + 3 + 2 < 15$ $x < 5$ so **x = 4**

