Advanced Programme Mathematics IEB

BOOK 1

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GRADE

10-12

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COMPULSORY
MODULES

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Advanced Programme Mathematics IEB

Book 1: Compulsory Modules

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**Marilyn Buchanan, Anne Eadie, Carl Fourie,
Noleen Jakins & Ingrid Zlobinsky-Roux**

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CONTENTS of Book 1: Compulsory Modules

IMPORTANT TO NOTE

Advanced Programme Mathematics is not an independent subject.

Knowledge and understanding of the core mathematics curriculum is a prerequisite as each module of the Advanced Mathematics Programme is introduced. In TAS AP study guides, we have not wanted to duplicate the development and mastering of core maths concepts where these are dealt with timeously in the core curriculum, as noted in the standard pace setters. Learners and teachers should therefore incorporate their core maths resources as part of their work for AP Maths.

TYPE ❷: $y = a|x| + q$

Observe the effects of both a and q in each case.

Worked Example 8

Draw the following graphs:

(a) $\boxed{y = 3|x| - 6}$ (b) $\boxed{y = -2|x| + 4}$

Solutions

Determine for each graph:

- The ***y*-intercept** (by substituting $x = 0$)

(a)	(b)
$y = 3\|0\| - 6 = 0 - 6 = -6$ $\therefore\ (0; -6)$	$y = -2\|0\| + 4 = 0 + 4 = 4$ $\therefore\ (0; 4)$

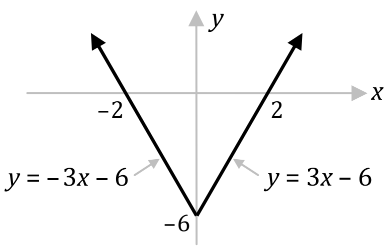
- The ***x*-intercept(s)** (by substituting $y = 0$)

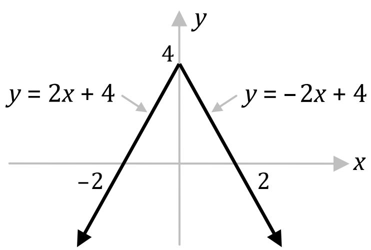
(a)	(b)
$3\|x\| - 6 = 0$ $\therefore\ 3\|x\| = 6$ $\therefore\ \|x\| = 2$ $\therefore\ x = \pm 2$	$-2\|x\| + 4 = 0$ $\therefore\ -2\|x\| = -4$ $\therefore\ \|x\| = 2$ $\therefore\ x = \pm 2$

$\therefore\ (-2; 0)$ & $(2; 0)$

- The **critical point**

(a)	(b)
The **minimum** y-value $= 3\|0\| - 6$ ... Min. of $\|x\| = 0$ $= -6$ $\therefore\ (0; -6)$	The **maximum** y-value $= -2\|0\| + 4$ ... Max. of $\|x\| = 0$ $= 4$ $\therefore\ (0; 4)$





Note the equations of the **left & right arms.**

Worked Example 9

Draw the following parabolas:

(a) $\boxed{y = 3x^2 - 6}$ (b) $\boxed{y = -2x^2 + 4}$

Solutions

Determine for each graph:

- The ***y*-intercept** (by substituting $x = 0$)

(a)	(b)
$y = 3(0)^2 - 6 = 0 - 6 = -6$ $\therefore\ (0; -6)$	$y = -2(0)^2 + 4 = 0 + 4 = 4$ $\therefore\ (0; 4)$

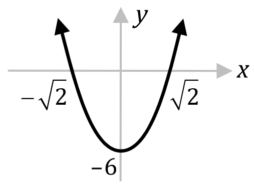
- The ***x*-intercept(s)** (by substituting $y = 0$)

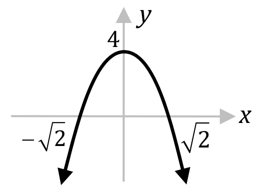
(a)	(b)
$3x^2 - 6 = 0$ $\therefore\ 3x^2 = 6$ $\therefore\ x^2 = 2$ $\therefore\ x = \pm\sqrt{2}$	$-2x^2 + 4 = 0$ $\therefore\ -2x^2 = -4$ $\therefore\ x^2 = 2$ $\therefore\ x = \pm\sqrt{2}$

$\therefore\ \left(-\sqrt{2}; 0\right)$ & $\left(\sqrt{2}; 0\right)$

- The **critical point**

(a)	(b)
The **minimum** y-value $= 3(0)^2 - 6$ ... Min. of $x^2 = 0$ $= -6$ $\therefore\ (0; -6)$	The **maximum** y-value $= -2(0)^2 + 4$ ... Max. of $x^2 = 0$ $= 4$ $\therefore\ (0; 4)$





TYPE ❷ parabolas: $\boldsymbol{y = ax^2 + q}$

Observe the similarities of the features and of the effects of a and q to those of the absolute value graphs in (*a*) and (*b*) in Worked Example 8.

Graphs	Shape	Range
(a)		$y \geq -6$
(b)		$y \leq 4$

Grade 10 Complex Numbers Exam Questions

(Solutions on p. 19 in the Answer book)

1. Factorise $x^3 - 1$, and hence solve $x^3 - 1 = 0$ for $x \in C$. *(IEB 2008)*

2. Calculate the values of a and b so that $\frac{a+3i}{2-5i}.bi = -11 - 13i$ *(IEB 2013)*

3. Given the complex numbers $z = 5 - 2i$ and $w = 6i - 1$.
   Determine in simplest form: $2z - iw$. *(IEB 2014)*

4. Determine, in terms of a and b, the real part of the complex expression $\frac{a+bi}{a-bi}$. *(IEB 2015)*

5. The quadratic equation $x^2 - 2x + p = 0$ has a root $x = q + \sqrt{3}i$.
   Find the rational values of p and q. *(IEB 2016)*

6. (a) It is given that $px^2 + px + 1 = 0$.
   Determine real values of p such that the solutions of the equation are of the form $a + bi$ where a and b are rational and $b \neq 0$.

   (b) Evaluate: $i + i^2 + i^3 + \ldots + i^{2017}$ *(IEB 2017)*

7. Thabo is practising division of complex numbers of the form $a + bi$, where $a, b \in \mathbb{R}$. He notices that:

   $\frac{3+2i}{-2+3i} = -i$, $\frac{5-7i}{7+5i} = -i$ and $\frac{4+5i}{-5+4i} = -i$.

   Prove that $\frac{a+bi}{-b+ai} = -i$ for all $a, b \in \mathbb{R}$. *(IEB 2018)*

8. Given that $m = 4 + 2i$ and $n = -2 - i$.
   Simplify the following expressions; show all calculations:

   (a) $m - 2n^*$ (b) $\frac{m}{n}$

9. (a) Factorise $x^2 + 8x + 25$ with complex numbers.

   (b) Find a quadratic equation that has a solution of $2 + 3i$.

10. Consider the following equation: $x^2 - 4x - 8 = 0$

    (a) Calculate the value of the discriminant.

    (b) Comment on the nature of the roots.

    (c) What constant must be added to the left hand side of the equation, so that the equation has one double real root? (Remember that 1 double root is the same as 2 equal roots.)

11. Write the complex number $w = -6 + 2i$ with polar coordinates.
    Now sketch the number in the Argand Plane.

12. Given that $z = -1 + 4i$, calculate the value of the following expressions. Show how these values are obtained and represented on the Argand plane.

    (a) $z.i^3$ (b) $z + 1$

    (c) $2z + z^*$ (d) $z.z^*$

13. Solve for p and q if $(3 + i)(p + qi) = -4 + 2i$.

14. If $a + bi$ is a root of the quadratic equation $x^2 + kx + t = 0$, use Vieta's Formulae to show that $a^2 + b^2 = t$ and $2a + k = 0$.

Worked Example 5

Decompose $\dfrac{3x^2 - x + 4}{(x + 1)(x - 1)^2}$ into partial fractions.

Solution

$$\frac{3x^2 - x + 4}{(x + 1)(x - 1)^2} = \frac{A}{(x + 1)} + \frac{B}{(x - 1)} + \frac{C}{(x - 1)^2}$$

We can go straight to the step where we equate numerators if we are able to do this.

$$\therefore\ 3x^2 - x + 4 \equiv A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)$$

Now substitute suitable values for *x*.

Let $x = -1$: $3 + 1 + 4 = A(4)$

$\therefore\ A = 2$

Let $x = 1$: $3 - 1 + 4 = C(2)$

$\therefore\ C = 3$

The value of *B* must now be found. We can choose any value for *x* and substitute all other values found.

Let $x = 0$: $4 = A - B + C$

$\therefore\ 4 = 2 - B + 3$

$\therefore\ B = 1$

$$\therefore\ \frac{3x^2 - x + 4}{(x + 1)(x - 1)^2} = \mathbf{\frac{2}{(x+1)} + \frac{1}{(x-1)} + \frac{3}{(x-1)^2}}$$

Exercise 5.3

(Solutions on p. 27 in the Answer book)

1. Decompose/resolve the following into their partial fractions.

(a) $\dfrac{3x + 4}{(x + 3)^2}$

(b) $\dfrac{x^2 - 7x + 12}{x(x - 2)^2}$

(c) $\dfrac{8x - 12}{(x + 3)(x^2 - 6x + 9)}$

(d) $\dfrac{2x^2 - 9x + 16}{x(x - 2)^2 + 1(x - 2)^2}$

(e) $\dfrac{2x^2 + 1}{x^2(x^2 - 2x + 1)}$

(f) $\dfrac{-x^2 + 9x - 27}{x(x - 3)^3}$

(g) $\dfrac{x}{(x - 3)^2}$

(h) $\dfrac{6 + 26x - x^2}{(2x - 1)(x + 2)^2}$

2. Note that in each of the following rational functions, the degree of the numerator is higher than (or equal to) the degree of the denominator.

First rewrite each function as $f(x) + \dfrac{g(x)}{\text{Denominator}}$ where the degree of $g(x)$ is lower than that of the denominator, then decompose the second term into Partial fractions.

We will revisit this skill in Chapter 17: Derivative Applications, when sketching rational functions.

(a) $\dfrac{x^3 - x + 2}{x^2 - 1}$

(b) $\dfrac{4x^2 - 14x + 2}{4x^2 - 1}$

(c) $\dfrac{10x^3 - 15x^2 - 35x}{x^2 - x - 6}$

Worked Example 9

Find possible functions for f and g such that $F = f \circ g$ given:

(a) $F(x) = (x^2 - 4)^3$ (b) $F(x) = \sqrt{x^3 - 1}$ (c) $F(x) = \dfrac{2}{2x - 2}$

Solutions

(a) $f(x) = x^3$ and $g(x) = x^2 - 4$ **or** $f(x) = (x - 4)^3$ and $g(x) = x^2$

(b) $f(x) = \sqrt{x}$ and $g(x) = x^3 - 1$ **or** $f(x) = \sqrt{x - 1}$ and $g(x) = x^3$

(c) $f(x) = \dfrac{2}{x}$ and $g(x) = 2x - 2$

Exercise 7.3

(Solutions on p. 37 in the Answer book)

1. If $f(x) = x^2$ and $g(x) = x - 3$, find:

   (a) $f(g(5))$ (b) $g(f(5))$ (c) $f(g(x))$

   (d) $g(f(x))$ (e) $f(f(x))$ (f) $g(g(x))$

2. Given that $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ and $g(x) = \sqrt{x} - 1$.

   Find the following, if possible.

   (a) $f(g(9))$ (b) $g(f(9))$ (c) $f(g(0))$

   (d) $g(f(0))$ (e) $f(g(-4))$ (f) $g(f(-4)$

   (g) $f(g(x))$ (h) $g(f(x))$

3. In each of the following, find $f \circ g$ and $g \circ f$, and state the domains.

   (a) $f(x) = 3x - 2$ and $g(x) = 2 - x$ (b) $f(x) = x^2$ and $g(x) = x + 2$

   (c) $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$ (d) $f(x) = x + 2$ and $g(x) = \sqrt{x^2}$

4. Given $f(x) = 2x - 6$ and $g(x) = \dfrac{1}{2}x + 3$.

   Is $f \circ g = g \circ f$? Explain your observation.

> **Note:** We noticed that in Question 4 there are **two equal composite functions.**
>
> In general $f \circ g \neq g \circ f$
>
> So when is $f \circ g = g \circ f$?
>
> This will occur when the function $g = f^{-1}$.
>
> ***NB: These are inverse functions and will be introduced after this exercise.***
>
> And then $f \circ g = g \circ f = x$

5. If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x + 1}$, is $f \circ g = g \circ f$? Explain.

6. If $F(x) = f \circ g$, find $f(x)$ and $g(x)$ in each of the following.

   From (a) to (e) there is more than one possible solution , but only give one solution.

   (a) $F(x) = 2(x + 3)^2 - 5(x + 3)$ (b) $F(x) = \dfrac{1}{5 - x}$

   (c) $F(x) = 5(\sin x)^3$ (d) $F(x) = \sqrt{1 - 3x}$

   (e) $F(x) = \tan 3x$

   In the following, give two possible options.

   (f) $F(x) = (x^2 - 1)^2$ (g) $F(x) = \left(\sqrt{x} + 2\right)^2 + \sqrt{x} + 2$

   (h) $F(x) = \sqrt{x^2 - 9}$ (i) $F(x) = \sqrt{3x - 1} + \dfrac{1}{\sqrt{3x - 1}}$

7. If $f(x) = \sqrt{x + 1}$, $g(x) = \dfrac{1}{x}$ and $h(x) = x + 1$, find:

   (a) $h \circ g \circ f(3)$ (b) $f \circ g \circ h(3)$ (c) $g \circ h \circ f(3)$

And, one other Absolute Value graph . . .

Worked Example 5

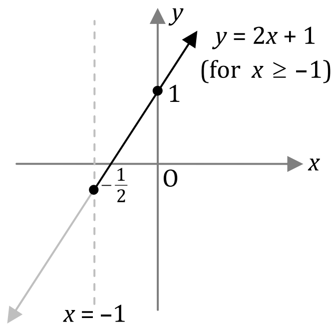
$$\boxed{y = -x + 3|x + 1| - 2}$$

Solution

For $x + 1 \geq 0$, i.e. $x \geq -1$:

$y = -x + 3(x + 1) - 2$

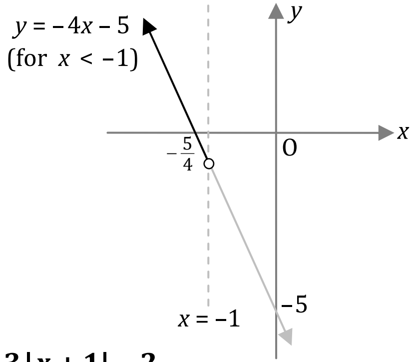
$\therefore\ y = 2x + 1$



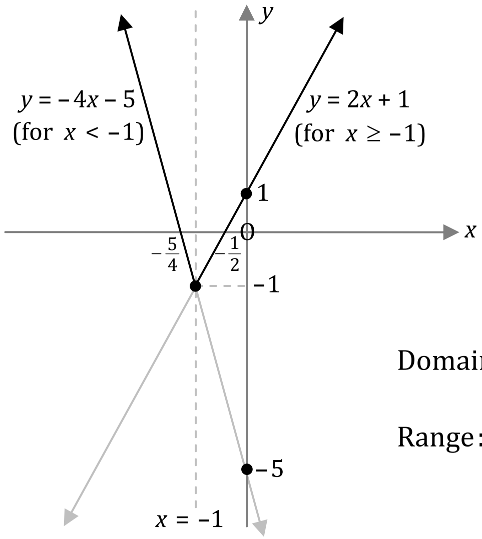
For $x + 1 < 0$, i.e. $x < -1$:

$y = -x + 3(-x - 1) - 2$

$\therefore\ y = -4x - 5$



So, the required graph: $y = -x + 3|x + 1| - 2$



Domain: $x \in \mathbb{R}$

Range: $y \geq -1$

The grey bits are not part of *this* graph. They indicate the invalid sections of the graphs.

Exercise 8.4

(Solutions on p. 47 in the Answer book)

1. Sketch the following graphs, indicating any intercepts with the axes, turning points, critical points and asymptotes. Determine the domain and range of each function.

   (a) $y = -|x|^2 + 2|x| + 15$

   (b) $y = \dfrac{5}{|x| + 1} - 2$

   (c) $y = 2(|x| - 1)^2 + 1$

   (d) $y = 3.2^{|x| - 1} - 2$

   (e) $y = |x - 3| + x + 1$

   (f) $y = 2|x + 1| - x + 5$

   (g) $y = x|x - 3| + 2$

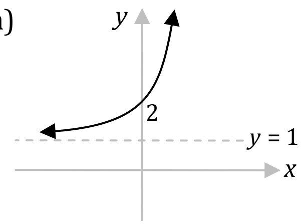
   (h) $y = \dfrac{x^2 - 25}{-|x + 5|}$

   (i) $y = \dfrac{-6}{|x - 1| + 2} + 1$

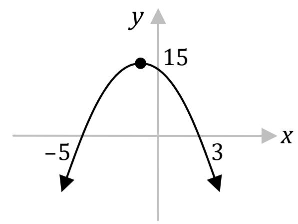
   (j) $y = 9 - 3^{|x + 1|}$

2. Given the following diagrams of $y = f(x)$, draw the diagrams of $y = f(|x|)$:

   (a)



   (b)



3. Sketch the graph of $y = \dfrac{|x|}{x}$

RULES FOR DERIVATIVES

1. The Constant rule

$f(x) = k$ where k is a constant, then $f'(x) = 0$.

2. The Power rule

$f(x) = x^n$ where $n \in \mathbb{R}$, then $f'(x) = nx^{n-1}$.

3. The Constant-Power rule

$D_x[k.f(x)] = k.f'(x)$

The derivative of a constant multiplied by a function is equal to the constant multiplied by the derivative of the function.

Thus we have:

$f(x) = x = x^1 \quad \Rightarrow \quad f'(x) = 1x^0 = 1$

$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x^1 = 2x$

$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$

$f(x) = \dfrac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(x) = -1x^{-2} = \dfrac{-1}{x^2}$

$f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

$f(x) = x^{\pi} \quad \Rightarrow \quad f'(x) = \pi x^{\pi - 1}$

4. The Sum (Difference) rule

$D_x[f(x) \pm g(x)] = f'(x) \pm g'(x)$

The derivative of a sum (difference) of functions is equal to the sum (difference) of the derivatives of the functions.

Worked Example 4

Find the derivatives of the following functions:

(a) $f(x) = 3x^2$

(b) $f(x) = -2x^3 + 5x - 3$

(c) $f(x) = \dfrac{5}{x} + \sqrt{x}$ (NB: first change the expression to powers of x)

(d) $f(x) = \dfrac{3 + x - 3x^2 + x^3}{x^3}$

Solutions

(a) $f'(x) = \mathbf{6x}$

(b) $f'(x) = \mathbf{-6x^2 + 5}$

(c) $f(x) = 5x^{-1} + x^{\frac{1}{2}}$,

$\therefore f'(x) = -5x^{-2} + \dfrac{1}{2}x^{-\frac{1}{2}} = \mathbf{\dfrac{-5}{x^2} + \dfrac{1}{2\sqrt{x}}}$

(d) $f(x) = 3x^{-3} + x^{-2} - 3x^{-1} + 1$

$\therefore f'(x) = -9x^{-4} - 2x^{-3} + 3x^{-2} = \mathbf{\dfrac{-9}{x^4} - \dfrac{2}{x^3} + \dfrac{3}{x^2}}$

Exercise 10.3

(Solutions on p. 63 in the Answer book)

1. Determine the derivatives of the following functions:

   (a) $f(x) = x^2 + 3$

   (b) $f(x) = 5x^2 + 2x$

   (c) $f(x) = 4x^2 - x + 7$

   (d) $f(x) = \sqrt{x} + 4$

   (e) $f(x) = 3x - \dfrac{1}{\sqrt{x}}$

   (f) $f(x) = x^3 - 6x^2 + 9x - 4$

   (g) $f(x) = \dfrac{x^3}{3} + x^2 - 5x + 1$

   (h) $f(x) = \dfrac{x^2 - 4x}{x}$

   (i) $f(x) = \dfrac{3x^2 + x - 1}{x}$

2. (a) Find $\dfrac{dy}{dx}$ given $y = 3x^3 + 5x^2 - 4x - 3$

   (b) Find $g'(x)$ given $g(x) = \dfrac{4x^2 - 1}{2x + 1}$

3. Find the following derivatives. Leave answers with positive exponents:

   (a) $D_x\left[x^2 - \dfrac{1}{x^3}\right]$

   (b) $\dfrac{d}{dx}\left(\dfrac{1 + x^{\frac{3}{2}}}{\sqrt{x}}\right)$

   (c) $D_t\left[\dfrac{\sqrt{t} - 3t}{\sqrt{t}}\right]$

   (d) $\dfrac{d}{ds}\left(\dfrac{2s - s^2 + 3s^3}{s^2}\right)$

   (e) $f'(x)$ if $f(x) = \dfrac{2x^3 - x^2 - 8x + 4}{x - 2}$

(b) $\cos\left(x - \frac{\pi}{6}\right) = \sin(2x)$

$\therefore\ \cos\left(x - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2} - 2x\right)$

$\therefore\ \left(x - \frac{\pi}{6}\right) = \pm\left(\frac{\pi}{2} - 2x\right) + 2\pi k,\ k \in \mathbb{Z}$

$x - \frac{\pi}{6} = \frac{\pi}{2} - 2x + 2\pi k$ or $x - \frac{\pi}{6} = -\frac{\pi}{2} + 2x + 2\pi k$

$\therefore\ 3x = \frac{2\pi}{3} + 2\pi k$ $\qquad$ $\therefore\ -x = -\frac{\pi}{3} + 2\pi k$

$\therefore\ x = \mathbf{\frac{2\pi}{9} + \frac{2\pi k}{3}}$ $\qquad$ $\therefore\ x = \mathbf{\frac{\pi}{3} + 2\pi k}$

NOTE
Dividing through the equation by -1, gives us $-2\pi k$ for the last term, but since k can be any integer, we write $+2\pi k$.

More equations

Worked Example 13

Find the general solutions of the following equations.

(a) $3\cos 3x = \sin 3x$ $\qquad$ (b) $2\sin^2 x + 3\sin x - 2 = 0$

Solutions

NOTE
Divide by $\cos 3x$ to simplify equation to $\tan 3x$ = Ratio

(a) $\frac{3\cos 3x}{\cos 3x} = \frac{\sin 3x}{\cos 3x}$

$\therefore\ \tan 3x = 3$

$\therefore\ 3x = 1{,}249045\ldots + \pi k,\ k \in \mathbb{Z}$

$\therefore\ x \approx \mathbf{0{,}416 + \frac{\pi k}{3}}$

(b) $2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$ $\qquad$ Factorise the trinomial

$\therefore\ \sin x = \frac{1}{2}$ or $\sin x = -2$ n/a $\qquad$ Range is $[-1; 1]$

$\therefore\ x = \mathbf{\frac{\pi}{6} + 2\pi k}$ or $x = \mathbf{\frac{5\pi}{6} + 2\pi k},\ k \in \mathbb{Z}$

Exercise 11.4

(Solutions on p. 67 in the Answer book)

Solve for x giving the general solution to the following equations.
Give answers in terms of π or correct to 3 decimal places, where necessary.

1. $2\tan\left(x - \frac{\pi}{12}\right) = 1{,}45$
2. (a) $3\sin\left(2x + \frac{\pi}{6}\right) = 1{,}5$ $\qquad$ (b) Hence solve for x if $x \in [-\pi; 2\pi]$
3. $\cos\left(3x - \frac{\pi}{36}\right) = -\cos\left(x + \frac{\pi}{36}\right)$
4. (a) $\sin\left(2x + \frac{\pi}{6}\right) = \sin x$ $\qquad$ (b) Hence solve for x if $x \in [-\pi; \pi]$
5. $\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = -\tan x$
6. (a) $\cos 2x = \sin(x)$ $\qquad$ (b) Hence solve for x if $x \in [-2\pi; \pi]$
7. $2\cos 2x = 1{,}3$
8. $\sin\left(3x - \frac{\pi}{12}\right) = -\sin 4x$
9. (a) $\cos 3x = -\sin\left(x + \frac{\pi}{18}\right)$ $\qquad$ (b) Hence solve for x if $x \in \left[-\frac{\pi}{3}; \pi\right]$
10. $\frac{3}{2}\tan 2x - 1{,}34 = 2$
11. $3\sin 2x = 2\cos 2x$
12. $2\sin x.\cos x - \sin x = 0$
13. $2\sin^2 x - \sin x = 1$
14. $\sin^2 x + \cos x = 1$

Worked Example 7

Prove that $a^n - b^n$ is divisible by $a - b$ for $n \in \mathbb{N}$

Solution

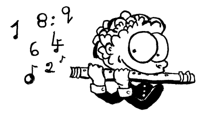
RTP: $a^n - b^n$ is divisible by $a - b$ for $n \in \mathbb{N}$ $\rightarrow$ A

Proof:

For $n = 1$:

$a^n - b^n = a^1 - b^1 = a - b$, which is divisible by $a - b$.

$\therefore$ A is true for $n = 1$.



Assume A is true for $n = k$, $k \in \mathbb{N}$

i.e. $a^k - b^k = p(a - b)$, $p \in \mathbb{N}$.

For $n = k + 1$:

$$\begin{aligned} a^n - b^n &= a^{k+1} - b^{k+1} \\ &= a^k.a - b^k.b \\ &= a^k.a - a.b^k + a.b^k - b^k.b && \text{Subtract and add } a.b^k \\ &= a(a^k - b^k) + b^k(a - b) && \text{Common factors of } a \text{ in first 2 terms and } (a-b) \text{ in the second 2 terms.} \\ &= a.p(a - b) + b^k(a - b) && \text{using the assumption} \\ &= (a - b)(ap + b^k) && \text{which is divisible by } (a - b). \end{aligned}$$

$\therefore$ If A is true for $n = k$, then it is also true for $n = k + 1$.

A is true for $n = 1$.

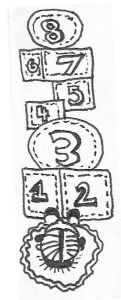
$\therefore$ By Mathematical Induction A is true for $n \in \mathbb{N}$.

Exercise 13.2

(Solutions on p. 78 in the Answer book)

Use Mathematical Induction to prove each of the following statements for all natural numbers.

1. $n^2 + n$ is an even number.
2. $n^3 + 2n$ is divisible by 3.
3. $6n^2 + 2n$ is divisible by 4.
4. $9^n - 4^n$ is divisible by 5.
5. $17^n - 7^n$ is divisible by 10.
6. $3^{2n} - 1$ is divisible by 8.
7. $7^n - 1$ is divisible by 6.
8. $3^{2n+4} - 2^{2n}$ is divisible by 5.
9. $5^{3n} - 2^{5n}$ is divisible by 31.
10. $3^{2n} + 7$ is divisible by 8.
11. $11^{n+1} + 12^{2n-1}$ is divisible by 133.
12. $a^{2n} - b^{2n}$ is divisible by $(a + b)$.
13. $8^n - 7n + 6$ is divisible by 7. *(IEB Exemplar 2008)*
14. $3^n + 3^{n+1} + 3^{n+2}$ is divisible by 13. *(IEB 2009)*



Worked Example 9

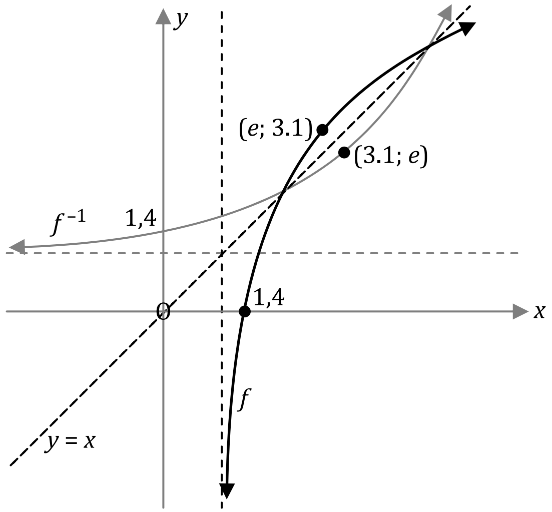
Find the equation of the inverse of $f(x) = 2\ln(x - 1) + 2$ and draw sketch graphs of $f(x)$ and $f^{-1}(x)$ on the same set of axes.

Solution

$$x = 2\ln(y - 1) + 2$$
$$\therefore\ x - 2 = 2\ln(y - 1)$$
$$\therefore\ \frac{x-2}{2} = \ln(y - 1)$$
$$\therefore\ y - 1 = e^{\frac{x-2}{2}}$$
$$\therefore\ y = e^{\frac{x-2}{2}} + 1$$

$$\boldsymbol{f^{-1}(x) = e^{\frac{x-2}{2}} + 1}$$

$$\boldsymbol{f(x) = 2\ln(x - 1) + 2}$$



Vertical asymptote:
$x = 1$

y-intercept (let $x = 0$):
$f(x) = 2\ln(0 - 1) + 2$
None

x-intercept (let $y = 0$):
$$0 = 2\ln(x - 1) + 2$$
$$\therefore\ x = 1{,}4$$

Create a point (let $x = e$):
$$y = 2\ln(e - 1) + 2$$
$$= 3{,}1 \quad (e; 3{,}1)$$

Domain: $x \in (1; \infty)$ **Range:** $y \in \mathbb{R}$

$$\boldsymbol{f^{-1}(x) = e^{x-2} + 1}$$

Horizontal asymptote: $y = 1$

y-intercept $= 1{,}4$

Create point: $(3{,}1; e)$

Domain: $x \in \mathbb{R}$ **Range:** $y \in (1; \infty)$

Exercise 14.7

(Solutions on p. 89 in the Answer book)

For each of the given functions:

1. $f(x) = -\ln(x - 3) - 1$
2. $f(x) = -e^{x+1} - 1$
3. $f(x) = 2e^{x} - 2$
4. $f(x) = 2\ln(x + 2) - 1$
5. $f(x) = \ln(x + 4) - 2$

(a) Find the equation of $f^{-1}(x)$.

(b) Find the domain and range of f and f^{-1}.

(c) Draw sketch graphs of f and f^{-1} on the same set of axes.

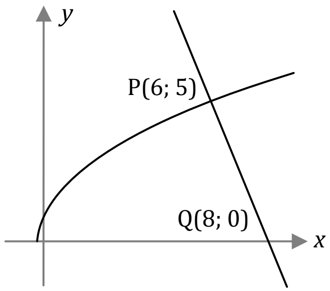
Notes & Exercises

Exercise 15.13

(Solutions on p. 103 in the Answer book)

1. The diagram shows a part of the curve $y = \sqrt{1 + 4x}$ and point P(6; 5) lying on the curve.

   The line PQ intersects the x-axis at Q(8; 0).

   (a) Show that PQ is a normal to the curve.

   (b) Determine the equation of the line PQ.

2. Determine the equation of the normal to the curve $3y^4 + 4x - x^2 \sin y - 4 = 0$ at the point (1; 0).

3. Consider the curve defined by $x^3 + y^3 - xy^2 = 5$.

   (a) Show that (1; 2) lies on the curve.

   (b) Determine an expression for $\frac{dy}{dx}$.

   (c) Hence, determine the equation of the normal to the curve at the point (1; 2). *(IEB 2014 Adapted)*

4. Consider the curve given by the equation $\sin y = x$ where $y \in \left[0; \frac{\pi}{2}\right]$.

   (a) Find an expression for $\frac{dy}{dx}$.

   (b) Hence, determine the equation of the normal to the curve where $x = 0{,}5$. *(IEB 2015)*

5. Determine the equation of the normal to the curve of $f(x) = e^{4x^2}$ at $x = \frac{1}{4}$. *(To two decimals)*

6. Show that the curves $y = e^x$ and $y = e^{-x}$ are orthogonal (curves intersect each other perpendicularly) at the point (0; 1).

Grade 12 Exam Questions

(Solutions on p. 104 in the Answer book)

1. (a) Given $f(x) = \frac{1}{1 - 2x}$, determine $f'(x)$ from first principles. *(IEB 2010)*

   (b) Given $f(x) = \frac{1}{\sqrt{2x + 1}}$, determine $f'(x)$ from first principles. *(IEB 2013 Adapted)*

2. Differentiate the following:

   (a) $f(x) = (3x - 2)(2x - 5)^5$ (b) $g(x) = \frac{3x^2 - 2x + 1}{5x - 1}$

3. Given that $\frac{d}{dx}\left[(2x + 3)^3(x - 4)\right] = (2x + 3)^2(ax + b)$, find the values of a and b.

4. (a) Determine $D_x\left[(x^3 + 1)^{\frac{3}{2}}\right]$.

   (b) Hence: given that $y = e^{3x}.(x^3 + 1)^{\frac{3}{2}}$, find the value of $\frac{dy}{dx}$ when $x = 0$.

5. Given that $f(x) = \ln(x^3 + 2)(x^2 + 3)$, find an expression for $f'(x)$.

6. Given that $f(x) = \ln(\sin 3x)$

   (a) Find an expression for the first derivative and simplify it.

   (b) Hence, find an expression for the second derivative.

7. Given that $f(x) = \tan x$

   (a) Determine $f''(x)$

   (b) Hence prove that $f''(x) - 2y = 2y^3$ *(IEB 2010)*

Exercise 17.9

(Solutions on p. 116 in the Answer book)

1. Draw neat sketch graphs of the following functions, showing all asymptotes, intercepts with axes and stationary points.

   (a) $f(x) = \dfrac{3x - 1}{x + 2}$ (b) $f(x) = \dfrac{1 - 2x}{x - 3}$

   (c) $f(x) = \dfrac{x^2 - 4}{x + 1}$ (d) $f(x) = \dfrac{x^2 + 2x + 1}{2x - 1}$

2. Given: $f(x) = \dfrac{-x^2 + x - 1}{x}$

   (a) Express $f(x)$ in asymptotic form i.e. $f(x) = q(x) + \dfrac{r(x)}{g(x)}$.

   (b) Calculate all turning points and asymptotes of f.

   (c) Sketch the curve of f.

3. Given: $f(x) = \dfrac{x^2 - x - 6}{x + 1}$

   (a) Determine the x-intercepts of f.

   (b) Determine the y-intercept of f.

   (c) Write down the equation of the vertical asymptote of f.

   (d) Determine the equation of the oblique asymptote of f.

   (e) Show that $f'(x) > 0$ for all values of x within the domain.

   (f) Hence sketch the graph of $y = f(x)$.

3 **Cubic expression in numerator and quadratic expression in the denominator**

3.1 In the form $y = \dfrac{(ax + b)(cx + d)(ex + f)}{(ex + f)(px + q)}$

Try to simplify first. If it can simplify, then same as quadratic over linear, but with a removable discontinuity at $x = -\dfrac{f}{e}$.

Worked Example 25

Draw a sketch graph of $f(x) = \dfrac{x^3 - 2x^2}{x^2 - x - 2}$.

Solution

Rational form: $f(x) = \dfrac{x^3 - 2x^2}{x^2 - x - 2}$ y-intercept: $(0; 0)$

Factorised form:
$$f(x) = \frac{x^2(x - 2)}{(x - 2)(x + 1)}$$
$$= \frac{x^2}{(x + 1)}; \quad x \neq 2$$

x-intercepts: $(0; 0)$. Two equal roots.

There is a removable discontinuity at $\left(2; \dfrac{4}{3}\right)$.

Asymptotic form:
$$f(x) = \frac{x(x + 1) - x}{x + 1}$$
$$= \frac{x(x + 1) - 1(x + 1) + 1}{x + 1} = (x - 1) + \frac{1}{x + 1}$$

CASE 8: Integration of rational functions with degree of numerator equal or one higher than degree of denominator.

In Chapter 17: Further Derivatives, we manipulated rational functions when considering asymptotes for graphs. This process can also be used in integration.

Worked Example 27

Given: $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$

We first manipulate the expression:

$$\frac{x^2+x+1}{x^2+1} = \frac{x^2+1+x}{x^2+1} = 1 + \frac{x}{x^2+1}$$

This can also be done using Long Division.

$$\int\left(1+\frac{x}{x^2+1}\right)dx = \int 1dx + \int \frac{x}{x^2+1}dx = x + \int \frac{2x}{2}\cdot\frac{1}{x^2+1}dx$$

$x^2+1 \geq 1 > 0$

$$= \boldsymbol{x + \frac{\ln(x^2+1)}{2} + c}$$

Worked Example 28

Given: $\displaystyle\int \frac{x^2+x-6}{x^2-5x+6}\,dx$

$$\frac{x^2+x-6}{x^2-5x+6} = \frac{(x+3)(x-2)}{(x-3)(x-2)} = \frac{x+3}{x-3},\ x \neq 2$$

$$= \frac{x-3+6}{x-3} = 1 + \frac{6}{x-3}$$

$$\int\left(1+\frac{6}{x-3}\right)dx = \boldsymbol{x + 6\ln|x-3| + c}$$

Worked Example 29

Given: $\displaystyle\int \frac{x^2-2}{x+1}\,dx$

Now the degree of the numerator is one more than that of the denominator.

$$\frac{x^2-2}{x+1} = \frac{x(x+1)-x-2}{x+1}$$

$$= \frac{x(x+1)-(x+1)+1-2}{x+1}$$

$$= x - 1 - \frac{1}{x+1}$$

We may prefer to do Long Division.

$$\int\left(x-1-\frac{1}{x+1}\right)dx$$

$$= \boldsymbol{\frac{x^2}{2} - x - \ln|x+1| + c}$$

Exercise 18.12

(Solutions on p. 131 in the Answer book)

In each of the following questions, first manipulate the expression before determining the integral.

1. $\displaystyle\int \frac{x-1}{x+1}dx$
2. $\displaystyle\int \frac{x+3}{x-5}dx$
3. $\displaystyle\int \frac{x^2-2x+3}{x}dx$
4. $\displaystyle\int \frac{x^3}{x^2-4}dx$
5. $\displaystyle\int \frac{x^2}{x-2}dx$
6. $\displaystyle\int \frac{x^2-5x+3}{x+2}dx$
7. $\displaystyle\int \frac{x^3}{x^2+1}dx$